Envariabelsanalys Z & TD, ht2017, Räkneövning 3.1

6.2.28 Beräkna $\int \frac{d\theta}{\cos\theta(1+\sin\theta)}$

Lösn.: $\int \frac{d\theta}{\cos\theta(1+\sin\theta)} = \int \frac{\cos\theta \, d\theta}{\cos^2\theta\,(1+\sin\theta)} =$

$$= \int \frac{\cos\theta \, d\theta}{(1-\sin^2\theta)(1+\sin\theta)} = \left\{ \begin{array}{l} x = \sin\theta. \\ dx = \cos\theta \, d\theta \end{array} \right\} =$$

$$= \int \frac{dx}{(1-x^2)(1+x)} = \int \frac{dx}{(1-x)(1+x)^2}$$

$$\frac{1}{(1-x)(1+x)^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{(1+x)^2} \iff$$

$$\iff 1 = A(1+x)^2 + B(1-x^2) + C(1-x)$$

$\underline{x=1:}$ $1 = A(1+1)^2 \iff A = 1/4$

$\underline{x=-1:}$ $1 = C(1+1) \iff C = 1/2$

$$1 = \frac{1}{4}(1+2x+x^2) + B - Bx^2 + \frac{1}{2} - \frac{1}{2}x \implies B = \frac{1}{4}$$

$$\therefore \int \frac{dx}{(1-x)(1+x)^2} = \int \left(\frac{1}{4}\frac{1}{1-x} + \frac{1}{4}\frac{1}{1+x} + \frac{1}{2}\frac{1}{(x+1)^2} \right) dx =$$

$$= -\frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| - \frac{1}{2}\frac{1}{x+1} = \{ x = \sin(\theta) \} =$$

$$= \frac{1}{4}\ln\left| \frac{1+\sin(\theta)}{1-\sin(\theta)} \right| - \frac{1}{2(1+\sin(\theta))} + C$$

6.3.2 Beräkna $\int \frac{x^2 \, dx}{\sqrt{1-4x^2}}$

Lösn.: $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-4x^2}} = \left\{ \begin{array}{l} x = \frac{1}{2}\sin(t),\ |t| \le \frac{\pi}{2} \\ dx = \frac{1}{2}\cos(t)\,dt \end{array} \right\} =$

$$= \int \frac{\frac{1}{4}\sin^2(t)}{\sqrt{1-\sin^2(t)}} \cdot \frac{1}{2}\cos(t)\,dt = \frac{1}{8}\int \frac{\sin^2(t)\cos(t)}{\cos(t)}\,dt =$$

$$= \frac{1}{8}\int \sin^2(t)\,dt = \frac{1}{8}\int \frac{1-\cos(2t)}{2}\,dt =$$

$$= \frac{1}{16}\left(t - \frac{\sin(2t)}{2}\right) = \frac{1}{16}\left(t - \sin(t)\cos(t)\right) =$$

$$= \frac{1}{16}\left(t - \sin(t)\sqrt{1-\sin^2(t)}\right) = \left\{ t = \arcsin(2x) \right\} =$$

$$= \frac{1}{16}\arcsin(2x) - \frac{1}{8}x\sqrt{1-4x^2} + C$$

6.3.20 Beräkna $\displaystyle\int \frac{x\,dx}{x^2-2x+3}$

Lösn.: $x^2-2x+3=0 \Rightarrow x = 1 \pm \sqrt{1-3}$ $\lightning$ Saknar reella rötter!

$$\int \frac{x\,dx}{x^2-2x+3} = \int \frac{x\,dx}{(x-1)^2+2} = \left\{ \begin{array}{l} t = x-1 \\ dt = dx \end{array} \right\} = \int \frac{t+1}{t^2+2}\,dt =$$

$$= \frac{1}{2}\int \frac{2t}{t^2+2}\,dt + \int \frac{1}{t^2+2}\,dt =$$

$$= \frac{1}{2}\ln|t^2+2| + \frac{1}{2}\int \frac{dt}{1+\left(\frac{t}{\sqrt{2}}\right)^2} =$$

$$= \frac{1}{2}\ln|t^2+2| + \frac{1}{2}\sqrt{2}\arctan\left(\frac{t}{\sqrt{2}}\right) = \left\{ t = x-1 \right\} =$$

$$= \frac{1}{2}\ln(x^2-2x+3) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C$$

6.5.17 Beräkna $\int_1^e \frac{dx}{x\sqrt{\ln(x)}}$ eller visa att integralen är divergent.

Lösn.: Problem med $x = 1$

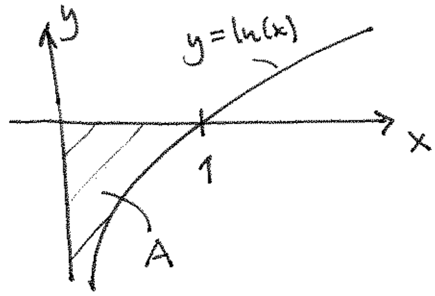
$$\int_{1+\varepsilon}^{e} \frac{dx}{x\sqrt{\ln(x)}} = \left\{ \begin{array}{l} t = \ln(x) \ . \ x = 1+\varepsilon \Leftrightarrow t = \ln(1+\varepsilon) \\ dt = \frac{1}{x}dx \ , \ x = e \Leftrightarrow t = 1 \end{array} \right\} =$$

$$= \int_{\ln(1+\varepsilon)}^{1} \frac{1}{\sqrt{t}}\,dt = 2\left[\sqrt{t}\right]_{\ln(1+\varepsilon)}^{1} =$$

$$= 2\left(1 - \sqrt{\ln(1+\varepsilon)}\right) \xrightarrow[\varepsilon \to 0^+]{} 2$$

$$\therefore \int_1^e \frac{dx}{x\sqrt{\ln(x)}} = 2$$

6.5.23 Beräkna arean av området under $y = 0$, över $y = \ln(x)$ och till höger om $x = 0$.



Lösn.: $A = \int_0^1 (0 - \ln(x))\,dx =$

$$= -\int_0^1 \ln(x)\,dx$$

Problem med $x = 0$.

$$\int_\varepsilon^1 \ln(x)\,dx = \left[x\ln(x)\right]_\varepsilon^1 - \int_\varepsilon^1 x \cdot \frac{1}{x}\,dx = -\varepsilon\ln(\varepsilon) - \left[x\right]_\varepsilon^1 =$$

$$= -\varepsilon\ln(\varepsilon) - 1 + \varepsilon$$

$$\lim_{\varepsilon \to 0^+} \varepsilon\ln(\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{\ln(\varepsilon)}{\frac{1}{\varepsilon}} = \{\text{l'Hopital}\} = \lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} =$$

$$= \lim_{\varepsilon \to 0^+} -\varepsilon = 0 \qquad (*)$$

$$\therefore \; A = \lim_{\varepsilon \to 0^+} -\int_{\varepsilon}^{1} \ln(x)\,dx = \lim_{\varepsilon \to 0^+} \left(\varepsilon \ln(\varepsilon) + 1 - \varepsilon \right) = \{ (*) \} =$$

$$= 1 \quad \text{a.e.}$$