Envariabelsanalys Z & TD, ht2017, Räkneövning 4.2

7.9.9 Lös ekvationen: $\dfrac{dy}{dt} = 2 + e^y$

Lösn.: $dy = (2+e^y)\,dt \Rightarrow \int \frac{1}{2+e^y}\,dy = \int dt = t + C$

$$\int \frac{1}{2+e^y}\,dy = \left\{\begin{array}{l} x = e^y \\ \ln(x) = y \\ dy = \frac{1}{x}\,dx \end{array}\right\} = \int \frac{dx}{x(x+2)}$$

$$\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \Leftrightarrow 1 = A(x+2) + Bx$$

$\underline{x=0}$: $1 = 2A \Leftrightarrow A = 1/2$

$\underline{x=-2}$: $1 = -2B \Leftrightarrow B = -1/2$

$$\int \frac{1}{x(x+2)}\,dx = \frac{1}{2}\int\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = \frac{1}{2}\left(\ln|x| - \ln|x+2|\right) =$$

$$= \frac{1}{2}\ln\left|\frac{x}{x+2}\right| = \{x = e^y\} = \frac{1}{2}\ln\left(\frac{e^y}{e^y+2}\right) = t + C$$

$$\Rightarrow \ln\left(\frac{e^y}{e^y+2}\right) = 2t + C \Leftrightarrow \frac{e^y}{e^y+2} = Ce^{2t} \Leftrightarrow$$

$$\Leftrightarrow e^y = Ce^{2t}e^y + 2Ce^{2t} \Leftrightarrow e^y(1 - Ce^{2t}) = 2Ce^{2t}$$

$$\Leftrightarrow e^y = \frac{2Ce^{2t}}{1 - Ce^{2t}} = \frac{2Ce^{2t}}{Ce^{2t}\left(\frac{e^{-2t}}{C} - 1\right)} = \frac{2}{Ce^{-2t} - 1}$$

$$\therefore y(x) = \ln\left(\frac{2}{Ce^{-2t}-1}\right) = -\ln\left(\frac{Ce^{-2t}-1}{2}\right) =$$

$$= -\ln\left(Ce^{-2t} - \frac{1}{2}\right)$$

7.9.15 Lös ekvationen: $\frac{dy}{dx} + y = x$

Lösn.: $\int 1\,dx = x \Rightarrow e^x$ integr. faktor

$$\Rightarrow e^x y'(x) + e^x y(x) = x e^x \Leftrightarrow \frac{d}{dx}\left(e^x y(x)\right) = x e^x$$

$$\Rightarrow e^x y(x) = \int x e^x\,dx = x e^x - e^x + C$$

$$\therefore\ y(x) = x - 1 + Ce^{-x}$$

7.9.20 Lös BVP: $\begin{cases} y'(x) + \cos(x)\,y(x) = 2x e^{-\sin(x)} \\ y(\pi) = 0 \end{cases}$

Lösn.: $\int \cos(x)\,dx = \sin(x) \Rightarrow$ Integr. faktor: $e^{\sin(x)}$

$$e^{\sin(x)} y'(x) + \cos(x) e^{\sin(x)} y(x) = 2x e^{-\sin(x)} \cdot e^{\sin(x)} = 2x$$

$$\Leftrightarrow \frac{d}{dx}\left(e^{\sin(x)} y(x)\right) = 2x \Rightarrow e^{\sin(x)} y(x) = \int 2x\,dx = x^2 + C$$

$$\therefore\ y(x) = x^2 e^{-\sin(x)} + Ce^{-\sin(x)}$$

$$y(\pi) = \pi^2 e^{-\sin(\pi)} + Ce^{-\sin(\pi)} = \pi^2 + C = 0$$

$$\Leftrightarrow C = -\pi^2$$

$$\therefore\ y(x) = e^{-\sin(x)}\left(x^2 - \pi^2\right)$$

7.9.22 Lös integralekvationen:

$$y(x) = 1 + \int_0^x \frac{(y(t))^2}{1+t^2}\,dt \qquad (*)$$

<u>Lösn.</u>: Derivera båda leden m.a.p. x :

$$y'(x) = \frac{d}{dx}\int_0^x \frac{(y(t))^2}{1+t^2}\,dt = \frac{(y(x))^2}{1+x^2} \Longrightarrow$$

$$\Rightarrow dy = \frac{y^2}{1+x^2}\,dx \Longrightarrow \int \frac{1}{y^2}\,dy = \int \frac{1}{1+x^2}\,dx$$

$$\Leftrightarrow -\frac{1}{y} = \arctan(x) + C \Leftrightarrow \frac{1}{y} = C - \arctan(x)$$

$$\therefore\ y(x) = \frac{1}{C - \arctan(x)}$$

Låt $x = 1$ i $(*)$:

$$y(0) = 1 + \int_0^0 \frac{(y(t))^2}{1+t^2}\,dt = 1$$

$$\Longrightarrow y(0) = \frac{1}{C - \arctan(0)} = \frac{1}{C-0} = 1 \Leftrightarrow C = 1$$

$$\therefore\ y(x) = \frac{1}{1 - \arctan(x)}$$