Envariabelsanalys Z&TD, ht2017, Räkneövning 5.1

3.7.8 Lös ODE:n $9y'' + 6y' + y = 0$

<u>Lösn.</u>: $9y'' + 6y' + y = 0 \iff y'' + \frac{2}{3}y' + \frac{1}{9}y = 0$

Karakteristisk ekvation: $r^2 + \frac{2}{3}r + \frac{1}{9} = 0$

$$\Rightarrow r = -\frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{1}{9}} = -\frac{1}{3}$$

$$\therefore \; y(x) = (C_1 + C_2 x)e^{-x/3}$$

3.7.12 Lös ODE:n $y'' + y' + y = 0$

<u>Lösn.</u>: Karak. ekv.: $r^2 + r + 1 = 0 \Rightarrow$

$$\Rightarrow r = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \; y(x) = e^{-x/2}\left(A\cos\left(\frac{\sqrt{3}x}{2}\right) + B\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

18.6.5 Lös ODE:n $y'' + 2y' + 5y = x^2$

<u>Lösn.</u>: (i) <u>Homogenlösn.</u>: Karak. ekv.: $r^2 + 2r + 5 = 0$

$$\Rightarrow r = -1 \pm \sqrt{1-5} = -1 \pm 2i$$

$$\Rightarrow y_h(x) = e^{-x}(A\cos(2x) + B\sin(2x))$$

(ii) <u>Partikulärlösn.</u>: Ansats: $y = Ax^2 + Bx + C$

$$\Rightarrow y' = 2Ax + B, \quad y'' = 2A$$

$$\Rightarrow y'' + 2y' + 5y = 2A + 2(2Ax + B) + 5(Ax^2 + Bx + C) =$$

$$= 5Ax^2 + (4A+5B)x + 2A + 2B + 5C \overset{\text{vill}}{=} x^2$$

$$\Rightarrow \begin{cases} 5A = 1 \Leftrightarrow A = 1/5 \\ 4A+5B = 0 \Leftrightarrow B = -\frac{4}{5}A = -\frac{4}{25} \\ 2A+2B+5C = 0 \Leftrightarrow C = -\frac{2}{5}A - \frac{2}{5}B = -\frac{2}{25} + \frac{8}{125} = -\frac{2}{125} \end{cases}$$

$$\Rightarrow y_p(x) = \frac{1}{5}x^2 - \frac{4}{25}x - \frac{2}{125} = \frac{1}{125}(25x^2 - 20x - 2)$$

$$\therefore y(x) = y_h(x) + y_p(x) = Ae^{-x}\cos(2x) + Be^{-x}\sin(2x) + \frac{1}{125}(25x^2 - 20x - 2)$$

18.6.10 Lös ODE:n $y'' + 2y' + 2y = e^{-x}\sin(x)$

Lösn.: (i) Homogen lösn.: Kar. ekv.: $r^2 + 2r + 2 = 0$

$$\Rightarrow r = -1 \pm \sqrt{1-2} = -1 \pm i$$

$$\Rightarrow y_h(x) = Ae^{-x}\cos(x) + Be^{-x}\sin(x)$$

(ii) Partikulärlösn.: Studera hjälpekvationen

$$u'' + 2u' + 2u = e^{-x} \cdot e^{ix} = e^{(-1+i)x}$$

Låt $u = ze^{(-1+i)x} \Rightarrow u' = z'e^{(-1+i)x} + (-1+i)ze^{(-1+i)x}$

$$u'' = z''e^{(-1+i)x} + 2(-1+i)z'e^{(-1+i)x} + (-1+i)^2 ze^{(-1+i)x} =$$

$$= (z'' + 2(-1+i)z' - 2iz)e^{(-1+i)x}$$

$$\Rightarrow u'' + 2u' + 2u = (z'' + 2(-1+i)z' - 2iz)e^{(-1+i)x} +$$

$$+ 2(z' + (-1+i)z)e^{(-1+i)x} + 2ze^{(-1+i)x} =$$

$$= (z'' + 2(-1+i)z' - 2iz + 2z' + 2(-1+i)z + 2z)e^{(-1+i)x} =$$

$$= (z'' + 2iz')e^{(-1+i)x} \overset{\text{vill}}{=} e^{(-1+i)x}$$

$$\Rightarrow z'' + 2iz' = 1 \quad \Rightarrow z_p = \frac{x}{2i} = -\frac{x}{2}i$$

$$\Rightarrow u_p = z_p e^{-x} \cdot e^{ix} = -\frac{x}{2}e^{-x} i \left(\cos(x) + i\sin(x)\right) =$$

$$= \frac{x}{2}e^{-x}\sin(x) - i\frac{x}{2}e^{-x}\cos(x)$$

$$\Rightarrow y_p = \mathrm{Im}(u_p) = -\frac{x}{2}e^{-x}\cos(x)$$

$$\therefore\ y(x) = y_h(x) + y_p(x) = Ae^{-x}\cos(x) + Be^{-x}\sin(x) - \frac{x}{2}e^{-x}\cos(x)$$

18.6.12 Lös ODE:n $y'' + 2y' + y = xe^{-x}$

<u>Lösn.</u>: (i) <u>Homogenlösn.:</u> Karak. ekv.: $r^2 + 2r + 1 = 0$

$$\Rightarrow r = -1 \pm \sqrt{1-1} = -1$$

$$\Rightarrow y_h(x) = (C_1 + C_2 x)e^{-x}$$

(ii) <u>Partikulärlösn.:</u> Låt $y = ze^{-x} \Rightarrow y' = z'e^{-x} - ze^{-x}$

$$y'' = z''e^{-x} - 2z'e^{-x} + ze^{-x} = (z'' - 2z' + z)e^{-x}$$

$$\Rightarrow y'' + 2y' + y = (z'' - 2z' + z)e^{-x} + 2(z' - z)e^{-x} + ze^{-x} =$$

$$= (z'' - 2z' + z + 2z' - 2z + z)e^{-x} = z''e^{-x} \overset{\text{vill}}{=} xe^{-x}$$

$$\Rightarrow z'' = x \Rightarrow z_p = \frac{x^3}{6} \Rightarrow y_p = z_p e^{-x} = \frac{1}{6}x^3 e^{-x}$$

$$\therefore\ y(x) = y_h(x) + y_p(x) = C_1 e^{-x} + C_2 x e^{-x} + \frac{x^3}{6}e^{-x}$$