Envariabelsanalys Z & TD, ht2017, Räkneövning 7.1

9.7.12 Beräkna arctan(0.2) med ett fel som är mindre än $5 \cdot 10^{-5}$.

Lösn.: $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$$\Rightarrow \arctan(0.2) = 0.2 - \frac{0.2^3}{3} + \frac{0.2^5}{5} - \frac{0.2^7}{7} + \dots$$

$$\frac{0.2^7}{7} = \frac{1}{7}\left(\frac{2}{10}\right)^7 = \frac{2^7}{7} \cdot 10^{-7} = \frac{128}{7} \cdot 10^{-7} < 5 \cdot 10^{-5}$$

$\Rightarrow$ Varje ytterligare term som adderas bidrar med termer som är mindre än $5 \cdot 10^{-5}$

$$\therefore \arctan(0.2) \approx 0.2 - \frac{0.2^3}{3} + \frac{0.2^5}{5} \quad (= 0.19740)$$

9.7.18 Beräkna Maclaurinutvecklingen för funktionen

$$L(x) = \int_0^x \cos(t^2)\, dt.$$

Lösn.: $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \forall x \in \mathbb{R}$

$$\Rightarrow \cos(t^2) = 1 - \frac{t^4}{2} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{4n}$$

$$\Rightarrow L(x) = \int_0^x \cos(t^2)\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{4n}\, dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^x t^{4n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!} x^{4n+1} =$$

$$= x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \dots$$

9.7.25 Beräkna, m.h.a. M-serier, gränsvärdet

$$\lim_{x\to 0} \frac{2\sin(3x) - 3\sin(2x)}{5x - \arctan(5x)}$$

Lösn.: $\dfrac{2\sin(3x) - 3\sin(2x)}{5x - \arctan(5x)} =$

$$= \frac{2\left(\cancel{3x} - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \ldots\right) - 3\left(\cancel{2x} - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \ldots\right)}{\cancel{5x} - \left(\cancel{5x} - \frac{5^3x^3}{3} + \frac{5^5 \cdot x^5}{5} - \ldots\right)} =$$

$$= \frac{-\frac{2\cdot 3^3}{3!}x^3 + \frac{2\cdot 3^5x^5}{5!} + \frac{3\cdot 2^3}{3!}x^3 - \frac{3\cdot 2^5}{5!}x^5 + \ldots}{\frac{5^3}{3}x^3 - \frac{5^5}{5}x^5 + \ldots} =$$

$$= \frac{\cancel{x^3}\left(\frac{3\cdot 8 - 2\cdot 27}{6} + \frac{2\cdot 3^5 - 3\cdot 2^5}{5!}x^2 + \ldots\right)}{\cancel{x^3}\left(\frac{125}{3} - 5^4x^2 + \ldots\right)} \underset{x\to 0}{\longrightarrow} \frac{-\frac{30}{6}}{\frac{125}{3}} =$$

$$= -\frac{30}{6}\cdot\frac{3}{125} = -\frac{3\cdot 5}{5\cdot 25} = -\frac{3}{25}$$

18.7.4 Lös BVP $\begin{cases} y'' + xy' + y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

Lösn.: Antag att $y(x) = \sum_{n=0}^{\infty} a_n x^n$. $|x| < R$

$$\Rightarrow y' = \sum_{n=0}^{\infty} a_n n x^{n-1} \Rightarrow xy' = \sum_{n=0}^{\infty} a_n n x^n$$

$$y'' = \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} =$$

$$= \left\{ \begin{matrix} k = n-2 \,.\, n=2 \Leftrightarrow k=0 \\ n = k+2 \,,\, n=\infty \Leftrightarrow k=\infty \end{matrix} \right\} = \sum_{k=0}^{\infty} a_{k+2}(k+1)(k+2)\,x^k =$$

$$= \{k=n\} = \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)\,x^n$$

$$\Rightarrow y'' + xy' + y = \sum_{n=0}^{\infty} \left(a_{n+2}(n+1)(n+2) + a_n \cdot n + a_n \right) x^n \overset{\text{vill}}{=} 0$$

$$\Rightarrow a_{n+2}(n+1)(n+2) + a_n(n+1) = 0 \qquad \forall n = 0, 1, \ldots$$

$$\Leftrightarrow a_{n+2} = -\frac{a_n \cdot \cancel{(n+1)}}{\cancel{(n+1)}(n+2)} = -\frac{a_n}{n+2}$$

$$y(0) = 1 \Leftrightarrow a_0 = 1, \quad y'(0) = 0 \Leftrightarrow a_1 = 0$$

$$a_2 = -\frac{a_0}{2} = -\frac{1}{2}, \quad a_3 = -\frac{a_1}{3} = 0$$

$$a_4 = -\frac{a_2}{4} = (-1)^2 \frac{1}{2 \cdot 4}, \quad a_5 = 0$$

$$a_6 = -\frac{a_4}{6} = (-1)^3 \cdot \frac{1}{2 \cdot 4 \cdot 6}, \quad a_7 = 0$$

$$a_8 = -\frac{a_6}{8} = (-1)^4 \cdot \frac{1}{2 \cdot 4 \cdot 6 \cdot 8}, \quad a_9 = 0$$

$$\vdots$$

$$a_{2n} = (-1)^n \cdot \frac{1}{2^n \cdot n!}, \quad a_{2n+1} = 0$$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} x^{2n} = \{t = x^2\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} \cdot t^n$$

$$\Rightarrow L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{2^{n+1}(n+1)!} \cdot \frac{2^n \cdot n!}{(-1)^n} \right| = \lim_{n\to\infty} \frac{1}{2(n+1)} = 0$$

$$\therefore \quad y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} x^{2n} \qquad \forall x \in \mathbb{R}$$