Lösningar tenta 10/1 -18, TMV138/181

1(a) $(2+\cos(x))y'(x)+\sin(x)y(x) = (2+\cos(x))^2 \iff$

$$\iff y'(x) + \frac{\sin(x)}{2+\cos(x)}y(x) = 2+\cos(x)$$

$$\int \frac{\sin(x)}{2+\cos(x)}dx = -\ln(2+\cos(x)) \Rightarrow$$

$$\Rightarrow \text{Integr. faktor}: e^{-\ln(2+\cos(x))} = \frac{1}{2+\cos(x)}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2+\cos(x)}y(x)\right) = 1 \Rightarrow$$

$$\Rightarrow \frac{1}{2+\cos(x)}y(x) = x + C$$

$$\therefore y(x) = (x+C)(2+\cos(x))$$

(b) $y' - e^y x = 3e^{y-x} \iff y' = 3e^y \cdot e^{-x} + e^y \cdot x$

$$\iff y' = e^y(3e^{-x}+x) \Rightarrow \int e^{-y}dy = \int (3e^{-x}+x)dx$$

$$\iff -e^{-y} = -3e^{-x} + \frac{x^2}{2} + C \iff$$

$$\iff e^{-y} = 3e^{-x} - \frac{x^2}{2} + C \iff$$

$$\iff -y = \ln\left(3e^{-x} - \frac{x^2}{2} + C\right)$$

$$\therefore y(x) = -\ln\left(3e^{-x} - \frac{x^2}{2} + C\right)$$

2(a) $\int \frac{x-1}{x^3+x}\,dx = \int \frac{x-1}{x(x^2+1)}\,dx$

$$\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \iff x-1 = A(x^2+1)+(Bx+C)x$$

$\underline{x=0}$: $-1 = A$

$$\Rightarrow Bx^2 + Cx = x^2 + \cancel{1} + x - \cancel{1} \Rightarrow \begin{cases} B = 1 \\ C = 1 \end{cases}$$

$$\Rightarrow \int \frac{x-1}{x(x^2+1)}\,dx = \int \left(\frac{x+1}{x^2+1} - \frac{1}{x} \right) dx =$$

$$= \frac{1}{2}\ln(x^2+1) + \arctan(x) - \ln|x| + C$$

(b) $\int_1^e \sin(\ln(x))\,dx = \left[x\sin(\ln(x))\right]_1^e - \int_1^e x\cos(\ln(x))\cdot\frac{1}{x}\,dx =$

$$= e\sin(1) - 1\cdot\sin(0) - \int_1^e \cos(\ln(x))\,dx =$$

$$= e\sin(1) - \left(\left[x\cos(\ln(x))\right]_1^e - \int_1^e x\cdot(-\sin(\ln(x)))\cdot\frac{1}{x}\,dx \right) =$$

$$= e\sin(1) - \left(e\cos(1) - 1\cdot\cos(0)\right) - \int_1^e \sin(\ln(x))\,dx =$$

$$= e\sin(1) - e\cos(1) + 1 - \int_1^e \sin(\ln(x))\,dx$$

$$\therefore \int_1^e \sin(\ln(x))\,dx = \frac{1}{2}e(\sin(1)-\cos(1)) + \frac{1}{2}$$

3 (a) $\dfrac{x(\sin(x)-\arctan(x))}{2e^{x^2}+\cos(2x)-3} =$

$$= \frac{x\left(\cancel{x}-\frac{x^3}{6}+\frac{x^5}{5!}-\ldots-\left(\cancel{x}-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right)\right)}{2\left(\cancel{1}+\cancel{x^2}+\frac{x^4}{2}+\frac{x^6}{3!}+\ldots\right)+\cancel{1}-\cancel{\frac{4x^2}{2}}+\frac{2^4x^4}{4!}-\frac{2^6x^6}{6!}+\ldots-\cancel{3}} =$$

$$= \frac{x\left(\left(\frac{1}{3}-\frac{1}{6}\right)x^3+\left(\frac{1}{5!}-\frac{1}{5}\right)x^5+\ldots\right)}{\left(1+\frac{2^4}{4!}\right)x^4+\left(\frac{1}{3}-\frac{2^6}{6!}\right)x^6+\ldots} =$$

$$= \frac{\cancel{x^4}\left(\frac{1}{6}+\left(\frac{1}{5!}-\frac{1}{5}\right)x^2+\ldots\right)}{\cancel{x^4}\left(\left(1+\frac{16}{24}\right)+\left(\frac{1}{3}-\frac{2^6}{6!}\right)x^2+\ldots\right)} \xrightarrow[x\to 0]{} \frac{\frac{1}{6}}{\frac{40}{24}} = \frac{5}{18}$$

(b) $\lim\limits_{x\to\infty} x\ln\left(\frac{x+1}{x-1}\right) = \lim\limits_{x\to\infty} x\ln\left(\frac{\cancel{x}(1+\frac{1}{x})}{\cancel{x}(1-\frac{1}{x})}\right) = \left\{\begin{array}{l} y=1/x \\ x\to\infty \Leftrightarrow y\to 0\end{array}\right\}$

$$= \lim_{y\to 0}\frac{1}{y}\ln\left(\frac{1+y}{1-y}\right) = \lim_{y\to 0}\frac{\ln(1+y)-\ln(1-y)}{y} =$$

$$= \lim_{y\to 0}\frac{y-\cancel{\frac{y^2}{2}}+\frac{y^3}{3}-\ldots-\left(-y-\cancel{\frac{y^2}{2}}-\frac{y^3}{3}-\ldots\right)}{y} =$$

$$= \lim_{y\to 0}\frac{2y+\frac{2y^3}{3}+\ldots}{y} = 2$$

4\. (i) <u>Homogenlösning</u>: Kar. ekv. : $r^2 + 2r - 3 = 0$

$$\Rightarrow r = -1 \pm \sqrt{1+3} = -1 \pm 2 \Rightarrow r_1 = 1, \; r_2 = -3$$

$$\Rightarrow y_h(x) = C_1 e^{x} + C_2 e^{-3x}$$

(ii) <u>Partikulärlösning</u>: Låt $y = z e^{x} \Rightarrow$

$$\Rightarrow y' = z'e^x + ze^x, \; y'' = z''e^x + 2z'e^x + ze^x$$

$$\Rightarrow y'' + 2y' - 3y = z''e^x + 2z'e^x + \cancel{ze^x} + 2z'e^x + \cancel{2ze^x} - \cancel{3ze^x} =$$

$$= z''e^x + 4z'e^x = (z'' + 4z')e^x \overset{\text{vill}}{=} (x+1)e^x$$

$$\Rightarrow z'' + 4z' = x + 1$$

Ansätt $z(x) = x(Ax + B) = Ax^2 + Bx$

$$\Rightarrow z' = 2Ax + B, \quad z'' = 2A$$

$$\Rightarrow z'' + 4z' = 2A + 8Ax + 4B = x + 1$$

$$\Rightarrow \begin{cases} 8A = 1 \iff A = 1/8 \\ 2A + 4B = 1 \Rightarrow B = 1/4 - A/2 = 3/16 \end{cases}$$

$$\Rightarrow z_p(x) = x\left(\frac{1}{8}x + \frac{3}{16}\right) \Rightarrow y_p = x\left(\frac{1}{8}x + \frac{3}{16}\right)e^x$$

$$\therefore \; y(x) = C_1 e^x + C_2 e^{-3x} + x\left(\frac{1}{8}x + \frac{3}{16}\right)e^x$$

5. (a) $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{2^{n+1}(n+1)} \Big/ \frac{1}{2^n \cdot n} =$

$$= \lim_{n\to\infty} \frac{\cancel{2^n} \cdot n}{\cancel{2^n} \cdot 2 \cdot (n+1)} = \lim_{n\to\infty} \frac{1}{2(1+\frac{1}{n})} = \frac{1}{2}$$

$$\Rightarrow R = \frac{1}{L} = 2 \Rightarrow |x-2| < 2 \Leftrightarrow$$

$$\Leftrightarrow -2 < x-2 < 2 \Leftrightarrow 0 < x < 4$$

$\underline{x=4:}$ $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot n}(4-2)^n = \sum_{n=1}^{\infty} \frac{2^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\underline{x=0:}$ $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot n}(-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (betingat) konv.

då $\left\{ \frac{(-1)^n}{n} \right\}$ alternerande och avtagande mot 0.

$\therefore$ Konv. intervall: $[0, 4)$

(b) $\underline{\text{Kvotkrit.:}}$ $\rho = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{5^{k+1}\sqrt{k+1}} \cdot \frac{5^k \sqrt{k}}{(-1)^k x^{2k}} \right| =$

$$= \lim_{k\to\infty} \frac{\cancel{5^k} \cdot \cancel{x^{2k}} \cdot x^2}{\cancel{5^k} \cdot 5 \cdot \cancel{x^{2k}}} \sqrt{\frac{k}{k+1}} = \lim_{k\to\infty} \frac{x^2}{5} \sqrt{\frac{1}{1+\frac{1}{k}}} = \frac{x^2}{5}$$

$$\rho < 1 \Leftrightarrow \frac{x^2}{5} < 1 \Leftrightarrow -\sqrt{5} < x < \sqrt{5}$$

$\underline{x = \pm\sqrt{5}:}$ $\sum_{k=1}^{\infty} \frac{(-1)^k}{\cancel{5^k}\sqrt{k}} \cancel{5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ (betingat) konv.
på samma sätt som (a)

$\therefore$ Konv. intervall: $[-\sqrt{5}, \sqrt{5}]$

6. Låt $y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow$

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$$

$$y''(x) = \sum_{n=1}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$\Rightarrow xy'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} = \left\{ \begin{matrix} k = n-1, \; n=2 \Leftrightarrow k=1 \\ n = k+1, \; n \to \infty \Leftrightarrow k \to \infty \end{matrix} \right\} =$$

$$= \sum_{k=1}^{\infty} a_{k+1} (k+1) k x^k = \{n = k\} = \sum_{n=1}^{\infty} a_{n+1} n(n+1) x^n$$

$$\Rightarrow xy'' - y = \sum_{n=1}^{\infty} a_{n+1} n(n+1) x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_{n+1} n(n+1) - a_n \right) x^n \overset{\text{vill}}{=} 0$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_{n+1} n(n+1) - a_n = 0 \qquad n = 1, 2, 3, \ldots \qquad (*) \end{cases}$$

$$y'(0) = 1 \Leftrightarrow a_1 = 1$$

$$(*) \Leftrightarrow a_{n+1} = \frac{a_n}{n(n+1)} \qquad n = 1, 2, 3, \ldots$$

$$a_2 = \frac{a_1}{1 \cdot 2} = \frac{1}{1 \cdot 2}, \quad a_3 = \frac{a_2}{2 \cdot 3} = \frac{1}{1 \cdot 2^2 \cdot 3}$$

$$a_4 = \frac{a_3}{3 \cdot 4} = \frac{1}{1 \cdot 2^2 \cdot 3^2 \cdot 4}, \quad a_5 = \frac{a_4}{4 \cdot 5} = \frac{1}{1 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5}$$

$$\vdots$$

$$a_n = \frac{1}{(n-1)!\,n!} \Rightarrow$$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!\,n!} x^n$$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0 \Rightarrow R = \infty$$

$$\therefore\ y(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!\,n!} x^n \quad \forall x \in \mathbb{R}$$

8 (b) <u>Bevis</u>: Från beviset av integralkrit. vet vi att

$$\int_1^n f(x)\,dx + f(n) \leq \sum_{k=1}^{n} f(k) \leq \int_1^n f(x)\,dx + f(1) \quad (*)$$

om f kont., positiv & avtagande då $x \geq 1$

Låt $f(x) = \frac{1}{n+x}$. $x \geq 1$, $n \in \mathbb{N}$

f kont., positiv & avtagande då $x \geq 1$, $n \in \mathbb{N}$

$$\int_1^n \frac{1}{n+x}\,dx = \Big[\ln|n+x|\Big]_1^n = \ln(2n) - \ln(n+1)$$

$$\Rightarrow \int_1^n f(x)\,dx + f(n) = \ln\left(\frac{2n}{n+1}\right) + \frac{1}{2n} \xrightarrow[n\to\infty]{} \ln(2)$$

$$\int_1^n f(x)\,dx + f(1) = \ln\left(\frac{2}{1+\frac{1}{n}}\right) + \frac{1}{n+1} \xrightarrow[n\to\infty]{} \ln(2)$$

Om vi låter $n \to \infty$ i $(*)$ får vi:

$$\ln(2) \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} \leq \ln(2)$$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \ln(2) \quad ∎$$