TMV138/181, ht2017, Övningstenta, Lösningar

1(a) $(1+x^2)y' + xy = x \iff y' + \frac{x}{1+x^2}y = \frac{x}{1+x^2}$

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2| = \frac{1}{2}\ln(1+x^2) \Rightarrow$$

$\Rightarrow$ Integ. faktor: $e^{\frac{1}{2}\ln(1+x^2)} = \sqrt{1+x^2}$

$$\Rightarrow \frac{d}{dx}\left(\sqrt{1+x^2}\,y\right) = \sqrt{1+x^2}\cdot\frac{x}{1+x^2} = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \sqrt{1+x^2}\,y = \int \frac{x}{\sqrt{1+x^2}}\,dx = \left\{\begin{array}{l} t = \sqrt{1+x^2} \\ x\,dx = t\,dt \end{array}\right\} =$$

$$= \int \frac{t\,dt}{t} = t + C = \sqrt{1+x^2} + C$$

$$\therefore\ y(x) = 1 + \frac{C}{\sqrt{1+x^2}}$$

(b) $(1+x^2)^2 y' = -x\sqrt{1+x^2}\,(1+y^2) \iff \frac{1}{1+y^2}y' = -\frac{x}{(1+x^2)^{3/2}}$

$$\Rightarrow \underbrace{\int \frac{1}{1+y^2}\,dy}_{=\arctan(y)} = -\int \frac{x}{(1+x^2)^{3/2}}\,dx$$

$$-\int \frac{x}{(1+x^2)^{3/2}}\,dx = \left\{\begin{array}{l} t = 1+x^2 \\ dt = 2x\,dx \end{array}\right\} = -\int \frac{1}{t^{3/2}}\cdot\frac{dt}{2} =$$

$$= -\frac{1}{2}\cdot\frac{t^{-1/2}}{-1/2} = \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{1+x^2}} + C$$

$$\Rightarrow \arctan(y) = \frac{1}{\sqrt{1+x^2}} + C$$

$\underline{y(0)=1}$: $\arctan(1) = \frac{1}{\sqrt{1+0}} + C \iff \frac{\pi}{4} = 1 + C$

$$\therefore\ y(x) = \tan\left(\frac{1}{\sqrt{1+x^2}} + \frac{\pi}{4} - 1\right)$$

2 (a) $\displaystyle\int \frac{\sin(2x)}{1+\sin(x)+\cos^2(x)}\,dx = \int \frac{2\sin(x)\cos(x)}{1+\sin(x)+1-\sin^2(x)}\,dx =$

$$= \left\{\begin{array}{l} t = \sin(x) \\ dt = \cos(x)dx \end{array}\right\} = \int \frac{2t}{2+t-t^2}\,dt = -2\int \frac{t}{t^2-t-2}\,dt =$$

$$= \left\{ t^2-t-2=0 \Rightarrow t = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = \frac{1}{2} \pm \frac{3}{2} \right\} =$$

$$= -2\int \frac{t}{(t-2)(t+1)}\,dt$$

$$\frac{t}{(t-2)(t+1)} = \frac{A}{t-2} + \frac{B}{t+1} \Leftrightarrow t = A(t+1) + B(t-2)$$

$\underline{t=2}$: $2 = 3A \Leftrightarrow A = 2/3$

$\underline{t=-1}$: $-1 = -3B \Leftrightarrow B = 1/3$

$$\Rightarrow -2\int \frac{t}{(t-2)(t+1)}\,dt = -2\int \left(\frac{2/3}{t-2} + \frac{1/3}{t+1}\right)dt =$$

$$= -2\left(\frac{2}{3}\ln|t-2| + \frac{1}{3}\ln|t+1|\right) = \{t = \sin(x)\} =$$

$$= -\frac{4}{3}\ln|\sin(x)-2| - \frac{2}{3}\ln|\sin(x)+1| + C$$

(b) $\displaystyle\int \ln(x^2+x+1)\,dx = \int 1\cdot\ln(x^2+x+1)\,dx = \{\text{p.i.}\} =$

$$= x\ln(x^2+x+1) - \int x\cdot\frac{2x+1}{x^2+x+1}\,dx$$

$$\begin{array}{r|l} & 2 \\ \hline x^2+x+1 & 2x^2+x \\ & -(2x^2+2x+2) \\ \hline & -x-2 \end{array}$$

$$\Rightarrow \int \frac{2x^2+x}{x^2+x+1}\,dx = \int\left(2 - \frac{x+2}{x^2+x+1}\right)dx = 2x - \int \frac{x+2}{x^2+x+1}\,dx$$

$$\int \frac{x+2}{x^2+x+1}\,dx = \int \frac{x+2}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \left\{\begin{array}{l} t = x+\frac{1}{2} \\ dt = dx \end{array}\right\} =$$

$$= \int \frac{t - \frac{1}{2} + 2}{t^2 + \frac{3}{4}}\, dt = \frac{1}{2} \int \frac{2t}{t^2 + \frac{3}{4}}\, dt + \frac{3}{2} \int \frac{1}{t^2 + \frac{3}{4}}\, dt =$$

$$= \frac{1}{2} \ln\left(t^2 + \frac{3}{4}\right) + 2 \int \frac{1}{1 + \left(\frac{2t}{\sqrt{3}}\right)^2}\, dt =$$

$$= \frac{1}{2} \ln\left(t^2 + \frac{3}{4}\right) + \sqrt{3} \arctan\left(\frac{2t}{\sqrt{3}}\right) = \left\{ t = x + \frac{1}{2} \right\} =$$

$$= \frac{1}{2} \ln\Big(\underbrace{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}_{x^2 + x + 1}\Big) + \sqrt{3} \arctan\left(\frac{2}{\sqrt{3}} x + \frac{1}{\sqrt{3}}\right)$$

$$\therefore \int \ln(x^2 + x + 1)\, dx = \left(x + \frac{1}{2}\right) \ln(x^2 + x + 1) - 2x +$$

$$+ \sqrt{3} \arctan\left(\frac{2}{\sqrt{3}} x + \frac{1}{2}\right) + C$$

3(a)
$$\frac{\cos(x)\,e^{x^2/2}-1}{x^2\arctan(x^2)}=\frac{\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots\right)\left(1+\frac{x^2}{2}+\frac{x^4}{4\cdot 2}+\dots\right)-1}{x^2\left(x^2-\frac{x^6}{3}+\frac{x^{10}}{5}-\dots\right)}=$$

$$=\frac{1+\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^4}{4!}+\dots-1}{x^4-\frac{x^8}{3}+\frac{x^{12}}{5}-\dots}=$$

$$=\frac{x^4\left(\frac{1}{8}-\frac{1}{4}+\frac{1}{4!}\right)+x^6(\dots)+\dots}{x^4-\frac{x^8}{3}+\frac{x^{12}}{5}-\dots}=$$

$$=\frac{\cancel{x^4}\left(\frac{3-6+1}{24}+x^2(\dots)+\dots\right)}{\cancel{x^4}\left(1-\frac{x^4}{3}+\frac{x^8}{5}-\dots\right)}\xrightarrow[x\to 0]{}-\frac{2}{24}=-\frac{1}{12}$$

(b)
$$\lim_{x\to\infty} xe^{-x^2}\int_0^x e^{t^2}\,dt=\lim_{x\to\infty}\frac{\int_0^x e^{t^2}\,dt}{\frac{e^{x^2}}{x}}=\{\text{l'Hopitals \&}\}=$$

$$=\lim_{x\to\infty}\frac{e^{x^2}}{\frac{2x^2e^{x^2}-e^{x^2}}{x^2}}=\lim_{x\to\infty}\frac{x^2e^{x^2}}{2x^2e^{x^2}-e^{x^2}}=$$

$$=\lim_{x\to\infty}\frac{x^2e^{x^2}}{x^2e^{x^2}\left(2-\frac{1}{x^2}\right)}=\frac{1}{2}$$

4. (i) Homogenlösn.: Kar. ekv.: $r^2 - 2r + 1 = 0 \Rightarrow$

$$\Rightarrow r = 1 \pm \sqrt{1-1} = 1$$

$$\Rightarrow y_h(x) = (C_1 + C_2 x)e^x$$

(ii) Partikulärlösn.: $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$$\Rightarrow y'' - 2y' + y = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

Studera hjälpekvationerna:

I. $v'' - 2v' + v = \frac{1}{2} \Rightarrow v_p = y_{p_1} = \frac{1}{2}$

II. $u'' - 2u' + u = -\frac{1}{2}e^{2ix}$

Låt $u = ze^{2ix} \Rightarrow u' = z'e^{2ix} + 2ize^{2ix}$

$$u'' = z''e^{2ix} + 4iz'e^{2ix} - 4ze^{2ix}$$

$$\Rightarrow u'' - 2u' + u = (z'' + 4iz' - 4z - 2z' - 4iz + z)e^{2ix} =$$

$$= (z'' + (-2+4i)z' - (3+4i)z)e^{2ix} \overset{\text{vill}}{=} -\frac{1}{2}e^{2ix}$$

$$\Rightarrow z_p = -\frac{1}{2}\cdot\frac{1}{(-3-4i)} = \frac{1}{6+8i}\cdot\frac{6-8i}{6-8i} = \frac{6}{100} - i\frac{8}{100}$$

$$\Rightarrow u_p = z_p e^{2ix} = \left(\frac{6}{100} - i\frac{8}{100}\right)\left(\cos(2x) + i\sin(2x)\right) =$$

$$= \frac{6}{100}\cos(2x) + \frac{8}{100}\sin(2x) + i\cdot(\ldots)$$

$$\Rightarrow y_{p_2} = Re(u_p) = \frac{1}{100}\left(6\cos(2x) + 8\sin(2x)\right)$$

$$\Rightarrow y_p = y_{p_1} + y_{p_2} = \frac{1}{2} + \frac{3}{50}\cos(2x) + \frac{4}{50}\sin(2x)$$

$$\Rightarrow y = y_h + y_p = (C_1 + C_2 x)e^x + \frac{1}{2} + \frac{3}{50}\cos(2x) + \frac{4}{50}\sin(2x)$$

(iii) Begynnelsevillkor:

$$y(0) = C_1 + \frac{1}{2} + \frac{3}{50} = 1 \iff C_1 = 1 - \frac{28}{50} = \frac{22}{50} = \frac{11}{25}$$

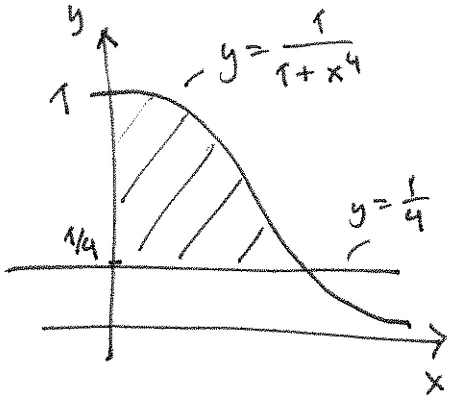
$$\Rightarrow y'(x) = \frac{11}{25}e^x + C_2e^x + C_2xe^x - \frac{6}{50}\sin(2x) + \frac{8}{50}\cos(2x)$$

$$y'(0) = \frac{11}{25} + C_2 + \frac{8}{50} = \frac{3}{5} + C_2 = 2 \iff C_2 = \frac{7}{5}$$

$$\therefore\ y(x) = \left(\frac{11}{25} + \frac{7}{5}x\right)e^x + \frac{1}{2} + \frac{3}{50}\cos(2x) + \frac{4}{50}\sin(2x)$$

5 (a) $\frac{1}{1+x^4} = \frac{1}{4} \iff 4 = 1 + x^4$

$\Rightarrow x = \overset{+}{(-)} 3^{1/4}$



$$V_y = \int_0^{3^{1/4}} 2\pi x \left(\frac{1}{1+x^4} - \frac{1}{4} \right) dx =$$

$$= 2\pi \int_0^{3^{1/4}} \frac{x}{1+x^4}\, dx - \frac{\pi}{2} \left[\frac{x^2}{2} \right]_0^{3^{1/4}} =$$

$$= \left\{ \begin{array}{ll} t = x^2 , & x = 0 \iff t = 0 \\ dt = 2x\,dx , & x = 3^{1/4} \iff t = \sqrt{3} \end{array} \right\} =$$

$$= \pi \int_0^{\sqrt{3}} \frac{1}{1+t^2}\, dt - \frac{\sqrt{3}\,\pi}{4} = \pi \left[\arctan(t) \right]_0^{\sqrt{3}} - \frac{\sqrt{3}\,\pi}{4} =$$

$$= \pi \arctan(\sqrt{3}) - \frac{\sqrt{3}\,\pi}{4} = \left\{ 2 \quad \sqrt{3} \quad 1 \right\} = \frac{\pi^2}{3} - \frac{\sqrt{3}\,\pi}{4} \quad \text{v.e.}$$

(b) $$L = \int_{\ln(3)/2}^{\ln(8)/2} \sqrt{1 + (f'(x))^2}\, dx = \int_{\ln(3)/2}^{\ln(8)/2} \sqrt{1 + e^{2x}}\, dx =$$

$$= \left\{ \begin{array}{ll} t = \sqrt{1+e^{2x}} & x = \frac{\ln(3)}{2} \iff t = \sqrt{1 + e^{\ln(3)}} = 2 \\ t^2 = 1 + e^{2x} & \\ x = \frac{1}{2}\ln(t^2 - 1) & x = \frac{\ln(8)}{2} \iff t = \sqrt{1 + e^{\ln(8)}} = 3 \\ dx = \frac{t}{t^2-1}\, dt & \end{array} \right\} =$$

$$= \int_2^3 t \cdot \frac{t}{t^2-1}\, dt = \int_2^3 \frac{t^2 - 1 + 1}{t^2 - 1}\, dt = \int_2^3 \left(1 + \frac{1}{t^2-1} \right) dt =$$

$$= 1 + \int_2^3 \frac{1}{(t-1)(t+1)}\, dt =$$

$$= \left\{ \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \iff 1 = A(t+1) + B(t-1) \right\} =$$

$$= 1 + \frac{1}{2} \int_2^3 \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt =$$

$$= 1 + \frac{1}{2} \Big[\ln|t-1| - \ln|t+1| \Big]_2^3 = 1 + \frac{1}{2} \Big[\ln \left| \frac{t-1}{t+1} \right| \Big]_2^3 =$$

$$= 1 + \frac{1}{2} \left(\ln\left(\frac{2}{4}\right) - \ln\left(\frac{1}{3}\right) \right) = 1 + \frac{1}{2} \ln\left(\frac{3}{2}\right) \quad \text{l.e.}$$

6 (a) $a_k = k\sin\left(\frac{1}{k}\right) = \dfrac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} \xrightarrow[k\to\infty]{} 1 \neq 0$

$$\Rightarrow \sum_{k=1}^{\infty} k\sin\left(\tfrac{1}{k}\right) \ \underline{\text{divergent}}$$

(b) $a_k = \sin\left(\frac{1}{k^2}\right)$, jämför med $\sum_{k=1}^{\infty} \frac{1}{k^2}$ dvs $b_k = \frac{1}{k^2}$

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k^2}\right)}{\frac{1}{k^2}} = 1$$

$$L = 1 < \infty \ \& \ \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \Rightarrow \sum_{k=1}^{\infty} \sin\left(\tfrac{1}{k^2}\right) \ \underline{\text{konvergent}}$$

(c) $a_n = \dfrac{n^2+3}{3n+2n^3}$, jämför med $\sum_{n=1}^{\infty} \frac{1}{n}$ dvs $b_n = \frac{1}{n}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n^2+3)}{3n+2n^3} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{3}{n^2}\right)}{\cancel{n^3}\left(2+\frac{3}{n^2}\right)} = \frac{1}{2}$$

$$L = \frac{1}{2} > 0 \ \& \ \sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{n^2+3}{3n+2n^3} \ \underline{\text{divergent}}$$

(d) Kvotkriteriet: $\rho = \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} =$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1)}{(n+1)\cdot n!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{n!} \ \underline{\text{divergent}}$$

(e) Kvotkriteriet: $\rho = \lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lim_{k\to\infty} \dfrac{((k+1)!)^2}{(2k+2)!} \cdot \dfrac{(2k)!}{(k!)^2} =$

$$= \lim_{k\to\infty} \frac{(k+1)^2 \cdot \cancel{(k!)^2} \cdot \cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}\,\cancel{(k!)^2}} = \lim_{k\to\infty} \frac{\cancel{k^2}\left(1+\frac{1}{k}\right)^2}{\cancel{k^2}\left(2+\frac{2}{k}\right)\left(2+\frac{1}{k}\right)} = \frac{1}{4} < 1$$

$$\Longrightarrow \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} \quad \underline{\text{konvergent}}$$

(f) Kvotkriteriet: $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{(n+1)^2}}{(n+1)!} \cdot \frac{n!}{2^{n^2}} =$

$$= \lim_{n\to\infty} \frac{2^{\cancel{n^2}+2n+1} \cdot \cancel{n!}}{(n+1)\cdot \cancel{n!} \cdot \cancel{2^{n^2}}} = \lim_{n\to\infty} \frac{2\cdot 4^n}{n+1} = \infty$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{2^{n^2}}{n!} \quad \underline{\text{divergent}}$$

8. Låt $f(x) = \frac{1}{x^2}\ln\left(1+\frac{1}{x}\right)$. Då f kont. och positiv då $x \geq 1$.

$$f'(x) = -2\cdot\frac{1}{x^3}\ln\left(1+\frac{1}{x}\right) + \frac{1}{x^2}\cdot\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right) =$$

$$= -\frac{1}{x^3}\left(2\ln\left(1+\frac{1}{x}\right) + \frac{1}{1+x}\right) \leq 0 \quad \text{då } x \geq 1$$

så f avtagande då $x \geq 1$.

Från beviset av integralkrit. följer att:

$$\sum_{k=1}^{n} f(k) \leq \int_1^n f(x)\,dx + f(1)$$

$$\Longleftrightarrow$$

$$\sum_{k=1}^{n} \frac{1}{k^2}\ln\left(1+\frac{1}{k}\right) \leq \int_1^n \frac{1}{x^2}\ln\left(1+\frac{1}{x}\right)dx + \ln(2)$$

$$\int_1^n \frac{1}{x^2}\ln\left(1+\frac{1}{x}\right)dx = \left\{\begin{array}{ll} t = \frac{1}{x} & x=1 \Leftrightarrow t=1 \\ dt = -\frac{1}{x^2}dx & x=n \Leftrightarrow t=1/n \end{array}\right\} =$$

$$= -\int_1^{1/n} \ln(1+t)\,dt = \int_{1/n}^{1} \ln(1+t)\,dt =$$

$$= \Big[t\cdot\ln(1+t)\Big]_{1/n}^{1} - \int_{1/n}^{1} t\cdot\frac{1}{1+t}\,dt =$$

$$= \ln(2) - \frac{1}{n}\ln\left(1+\frac{1}{n}\right) - \Big[t\Big]_{1/n}^{1} + \Big[\ln(1+t)\Big]_{1/n}^{1} =$$

$$= 2\ln(2) - 1 - \frac{1}{n}\ln\left(1+\frac{1}{n}\right) + \frac{1}{n} - \ln\left(1+\frac{1}{n}\right) \xrightarrow[n\to\infty]{} 2\ln(2) - 1$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2}\ln\left(1+\frac{1}{k}\right) \leq 2\ln(2) - 1 + \ln(2) = 3\ln(2) - 1$$

$\blacksquare$