Answers to even numbered exercises

1.1

20. $h \neq -4$.
22. $h = 6$.
24. False, true, false, true.
34. $T_1 = 20$, $T_2 = 27.5$, $T_3 = 30$, $T_4 = 22.5$.

1.2

20 (a) $h = -6, k \neq 2$ **(b)** $h \neq -6$, any k **(c)** $h = -6$, $k = 2$.
22. True, false (could be a row of zeroes in the coefficient matrix), false, true, false.
24. Not necessarily. The third row could consist entirely of zeroes and the second row entirely of zeroes on the left, with a pivot (i.e.: non-zero number) on the right.
26. Yes, since there cannot then be a row of zeroes in the coefficient matrix.
28. There should be a pivot in each column of the coefficient matrix, but not in the right-hand column of the augmented matrix.
30. $x + y + z = 1$ and $x + y + z = 2$.
34. Matlab exercise. We seek the interpolating polynomial

$$F(v) = a_o + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + a_5 v^5.$$

From the six given data points, we obtain the system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 1 & 4 & 4^2 & 4^3 & 4^4 & 4^5 \\ 1 & 6 & 6^2 & 6^3 & 6^4 & 6^5 \\ 1 & 8 & 8^2 & 8^3 & 8^4 & 8^5 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 2.9 \\ 14.8 \\ 39.6 \\ 74.3 \\ 119 \end{bmatrix}.$$

Plugging into Matlab, using short format and running the command "$> A\backslash\boldsymbol{b}$", I obtained the following solution correct to 4 decimal places :

$$a_0 = 0, \quad a_1 \approx 1.7125, \quad a_2 \approx -1.1948, \quad a_3 \approx 0.6615, \quad a_4 \approx -0.0701, \quad a_5 \approx 0.0026.$$

Hence, when $v = 750$, we estimate

$$F(750) \approx 0 + (1.7125)(750) - (1.1948)(750^2) + (0.6615)(750^3) \\ -(0.0701)(750^4) + (0.0026)(750^5) \approx 5.9509 \times 10^{11} \text{ lb.}$$

REMARK : Rounding off to 4 decimal places seems to introduce non-negligible errors. For example, if we plug in $v = 10$, then we get $F(10) \approx 118.145$, instead of the measured value of 119. For greater precision, one should run one's program in long format (16 decimal places).

1.3

2. $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$, $\boldsymbol{u} - 2\boldsymbol{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.
24. False, true, true, false, false.

1.4

16. The echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & -2 & -1 & | & b_1 \\ 0 & -2 & -2 & | & b_2 + 2b_1 \\ 0 & 0 & 0 & | & 6b_1 + 7b_2 + 2b_3 \end{bmatrix}.$$

Thus there is a solution if and only if $6b_1 + 7b_2 + 2b_3 = 0$.
24. True, true, true, false, true, false.

1.5

8. We want to write in parametric vector form the set of all $\boldsymbol{x} = [x_1\ x_2\ x_3\ x_4]^T \in \mathbb{R}^4$ such that $A\boldsymbol{x} = \mathbf{0}$. We are told that A is row equivalent to the given matrix, which we call U, i.e.:

$$U := \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}.$$

This row equivalence implies that U is an echelon form of A, and hence that $A\boldsymbol{x} = \mathbf{0}$ has the same solution set as $U\boldsymbol{x} = \mathbf{0}$. So we can proceed immediately to back substitution. The variables x_3 and x_4 are free, and we obtain

$$x_2 = -2x_3 + 4_4,$$
$$x_1 - 3(-2x_3 + 4x_4) - 8x_3 + 5x_4 = 0 \Rightarrow x_1 = 2x_3 + 7x_4.$$

Hence, the general solution to $A\boldsymbol{x} = \mathbf{0}$ in *parameter form* is

$$x_1 = 2x_3 + 7x_4, \quad x_2 = -2x_3 + 4x_4, \quad x_3, x_4 \in \mathbb{R}.$$

In order to express this is in *parametric vector form* we write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the solution set to $A\boldsymbol{x} = \mathbf{0}$ is given, in parametric vector form, by

$$\left\{ x_3 \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}.$$

A more compact way of writing this set is as

$$\text{Span}\{[2 \ -2 \ 1 \ 0]^T, [7 \ 4 \ 0 \ 1]^T\}.$$

OBS! The advantage of writing the solution set in parametric *vector form* is that it makes it clearer what the solution set "looks like". In the above example, it is a plane through the origin in $\mathbb{R}^4$, spanned by the vectors $(2, -2, 1, 0)$ and $(7, 4, 0, 1)$. This way of writing things will also prove useful when we get to Sections 4.2, 4.3 and 4.6, for answering a question like "Find a basis for the nullspace of the matrix A".

24. False, false, true, true, true.
26. Every $\boldsymbol{x} \in \mathbb{R}^3$ is a solution.

1.7

22. True, true, false, false.
24. The echelon form must be upper triangular, with non-zero entries on the main diagonal.
26. The given conditions imply that $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent. Then, in the echelon form, the fourth row will consist entirely of zeroes, and the first three rows will look as in Q.24.

28. Four (if there were a row of zeroes in the echelon form of the matrix, which we'll call A, then there would be no solution to $A\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{b}$).
30. n.
34. Well, $\boldsymbol{v}_1$ could be the zero vector. If not, the statement will be true.
36. Not necessarily, as $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ could already be linearly dependent. If not, then the statement will be true (see Q.26).
38. True. Any subset of a linearly independent set of vectors is linearly independent. Equivalently, any superset of a linearly dependent set of vectors is linearly dependent.

1.8

22. True, true, false (it's an 'existence' question), true, true.
32. To prove that a transformation T is not linear, it suffices to exhibit ONE example of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and a pair of scalars c_1, c_2 such that

$$T(c_1\boldsymbol{x} + c_2\boldsymbol{y}) \neq c_1T(\boldsymbol{x}) + c_2T(\boldsymbol{y}). \tag{1}$$

In the present case, take for example

$$\boldsymbol{x} = (0, 1), \ \boldsymbol{y} = (0, -1), \ c_1 = c_2 = 1.$$

Then

$$c_1\boldsymbol{x} + c_2\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{y} = (0, 1) + (0, -1) = (0, 0),$$

so, by (1), we need to show that

$$T(0, 0) \neq T(0, 1) + T(0, -1). \tag{2}$$

But according to the given formula for T, one has

$$T(0, 0) = (0, 0), \ \ T(0, 1) = (-2, -4), \ \ T(0, -1) = (-2, 4).$$

Hence the VL of (2) is just $(0, 0)$, whereas the HL is $(-4, 0)$, so VL $\neq$ HL, as desired.

34. To prove that a transformation T is linear, one must show that, for EVERY possible choice of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and a pair of scalars c_1, c_2,

$$T(c_1\boldsymbol{x} + c_2\boldsymbol{y}) = c_1T(\boldsymbol{x}) + c_2T(\boldsymbol{y}). \tag{3}$$

So let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, so that

$$c_1\boldsymbol{x} + c_2\boldsymbol{y} = (c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3).$$

According to the given formula for T, one has on the one hand that the VL of (3) is computed as

$$\begin{aligned} T(c_1\boldsymbol{x} + c_2\boldsymbol{y}) &= T(c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3) \\ &= (c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, -c_1x_3 - c_2y_3). \end{aligned} \tag{4}$$

On the other hand, the HL of (3) is computed as

$$\begin{aligned} c_1T(\boldsymbol{x}) + c_2T(\boldsymbol{y}) = c_1T(x_1, x_2, x_3) + c_2T(y_1, y_2, y_3) &= c_1(x_1, x_2, -x_3) + c_2(y_1, y_2, -y_3) \\ &= (c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, -c_1x_3 - c_2y_3). \end{aligned} \tag{5}$$

From (4) and (5), we see that the VL and HL of (3) coincide, v.s.v.

1.9

4. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

24. False (should say "into"), true, true, false (the statement just means T is a function), false (there are 4 possible transformations, and they are respectively : (i) the identity ($a = d = 1$), (ii) reflection in the horizontal axis ($a = 1, d = -1$), (iii) reflection in the vertical axis ($a = -1, d = 1$), (iv) reflection in the origin ($a = d = -1$)).

26. As shown in class, the standard matrix for T is the 2×3 matrix

$$A = A_T = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}.$$

For T to be injective, we require that the system $A\boldsymbol{x} = \mathbf{0}$ have only the trivial solution $\boldsymbol{x} = \mathbf{0}$. But the row operation $R_2 \mapsto R_2 - 4R_1$ reduces A to the echelon form $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 17 & -20 \end{bmatrix}$. Here we have a free variable, so $A\boldsymbol{x} = \mathbf{0}$ will have infinitely many solutions. Thus T is not injective.

32. m (see Theorem 12).

2.1

16. True, false (true without the + signs), true, false, true.
22. In general, the columns of an $m \times n$ matrix M are linearly dependent if and only if there is a non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $M\boldsymbol{x} = \mathbf{0}$.

So suppose the columns of B are linearly dependent. Thus there exists a non-zero vector $\boldsymbol{x}$ such that $B\boldsymbol{x} = \mathbf{0}$. Multiply both sides of this equation on the left by A, and we have $A(B\boldsymbol{x}) = A \cdot \mathbf{0} = \mathbf{0}$. But matrix multiplication is associative, so $A(B\boldsymbol{x}) = (AB)\boldsymbol{x}$. Thus $(AB)\boldsymbol{x} = \mathbf{0}$ so, by the same reasoning as before, the columns of AB must be linearly dependent.
24. Denote the columns of A by $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$. Since these span $\mathbb{R}^3$, there exist scalars $a_1, ..., a_n$, $b_1, ..., b_n$ and $c_1, ..., c_n$ such that

$$\sum_{i=1}^{n} a_i \boldsymbol{v}_i = \boldsymbol{e}_1, \quad \sum_{i=1}^{n} b_i \boldsymbol{v}_i = \boldsymbol{e}_2, \quad \sum_{i=1}^{n} c_i \boldsymbol{v}_i = \boldsymbol{e}_3.$$

Let

$$\boldsymbol{a} := [a_1 \; a_2 \cdots a_n]^T, \quad \boldsymbol{b} := [b_1 \; b_2 \cdots b_n]^T, \quad \boldsymbol{c} := [c_1 \; c_2 \cdots c_n]^T.$$

and let D be the $n \times 3$ matrix with these as its columns, i.e.: $D := [\boldsymbol{a} \; \boldsymbol{b} \; \boldsymbol{c}]$. Then $AD = I_3$ by design.
26. Let $\boldsymbol{b}$ be given and multiply both sides of the equation $AD = I_m$ on the right by $\boldsymbol{b}$. This yields $(AD)\boldsymbol{b} = \boldsymbol{b}$. By associativity of matrix multiplication, the left-hand side of this equals $A(D\boldsymbol{b})$. But then we have indeed a solution to $A\boldsymbol{x} = \boldsymbol{b}$, namely $\boldsymbol{x} = D\boldsymbol{b}$.

2.2

10. False (rather they reduce I_n to A^{-1}), true, false (should say *reverse* order), true, true.
12. We have $AD = I$. Left-multiply both sides by A^{-1} to get $A^{-1}(AD) = A^{-1}I$. On the one hand, $A^{-1}I = A^{-1}$, by definition of the identity matrix. On the other hand, since matrix multiplication is associative, we have $A^{-1}(AD) = (A^{-1}A)D = ID = D$. Hence, $D = A^{-1}$, as required.
18. Right-multiply both sides by B^{-1} to get $(AB)B^{-1} = (BC)B^{-1}$. By associativity, the left-hand side equals $A(BB^{-1}) = AI = A$. Hence, $A = BCB^{-1}$.
32. The matrix is not invertible, since the row operations $R_2 \mapsto R_2 + 4R_1$, $R_3 \mapsto R_3 + 2R_1$, $R_3 \mapsto R_3 + 2R_2$ take it to the echelon form $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

2.3

12. True, false (true with "onto"), true, false, false (to make the statement true you need an extra hypothesis, for example (i) that A be invertible, or (ii) that the equation be consistent for *every* $\boldsymbol{b}$).
16. Because A^T is also invertible, namely $(A^T)^{-1} = (A^{-1})^T$. See Theorem 2.2.6(c) and Theorem 2.3.8(e),(l).
24. The hypothesis implies that G is not invertible (Theorem 2.3.8(b)), hence implies that the columns of G are linearly dependent (Theorem 2.3.8(e)).

2.8

20. Can't be a basis, since there are 4 vectors there and $\mathbb{R}^3$ is 3-dimensional.
22. False, false (rather the corresponding columns in A form a basis for Col(A)), true, false, false.

2.9

18. True, false (it's the number of *free* variables), true, true, true.
20. Use Theorem 2.9.14 (same as Theorem 4.6.14). We are told that $n = 8$ and that dim(Nul(A)) = 3. Hence, rank(A) = 5.
22. Use the Basis Theorem (Theorem 2.9.15 or 4.5.12). Since $\mathbb{R}^5$ has dimension 5, if the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_5$ were linearly independent, then they would span the whole of $\mathbb{R}^5$, not just a 4-dimensional subspace.
24. The rank will equal one if every row is a multiple of every other. Here's an example : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$.
26. If rank(A) = 5, then dim(Col(A)) = 5 so, by the Basis Theorem again, any five linearly independent columns must span the whole column space.

3.1

20. The row operation is $R_2 \mapsto kR_2$. The determinant gets multiplied by k.
40. False, False (true if we replace 'sum' by 'product').

3.2

28. True, False, False, False.
32. $\det(rA) = r^n(\det A)$.

3.3

26. A typical vector $\boldsymbol{v}$ in the set $\boldsymbol{p} + S$ is of the form $\boldsymbol{v} = \boldsymbol{p} + \boldsymbol{s}$, for some vector $\boldsymbol{s} \in S$. Applying T and using linearity we have $T(\boldsymbol{v}) = T(\boldsymbol{p} + \boldsymbol{s}) = T(\boldsymbol{p}) + T(\boldsymbol{s})$, which is just a typical element of $T(\boldsymbol{p}) + T(S)$ since, by definition, $T(S) = \{T(\boldsymbol{s}) : \boldsymbol{s} \in S\}$.
32. Let T_1, T_2 be the names of the tetrahedra with sides $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ respectively. By the formula for the volume of a tetrahedron given in the text, we have that $\text{Vol}(T_1) = 1/6$, since it has perpendicular height one and its base is an equilateral triangle of side-length one, thus of area $1/2$.

Now the linear transformation defined by $T(\boldsymbol{e}_i) = \boldsymbol{v}_i$, $i = 1, 2, 3$ transforms T_1 to T_2. By definition, the matrix of this transformation is $M_T = [\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3]$, i.e.: the 3×3 matrix whose columns are the $\boldsymbol{v}$-vectors. By the geometric definition of determinant, we have that $\text{Vol } T_2 =$
$|\det M_T| \cdot$ (Vol T_1). Thus, by what we noted at the outset, it follows that

$$\text{Vol}(T_2) = \pm\frac{1}{6}\begin{vmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ | & | & | \end{vmatrix},$$

the sign depending on whether the determinant is positive or negative.

4.1

4. I will try so say this in words. Draw any line $\mathcal{L}$ in the plane not passing through $(0,0)$. Pick any point P on the line and let $\boldsymbol{v}$ be the vector $\vec{OP}$. Consider $2\boldsymbol{v}$. This is the vector $\vec{OQ}$, where Q is the point along the line through O and P, which is twice as far away from O as is P and in the same direction. Clearly, this point is not on your line $\mathcal{L}$, thus proving that $\mathcal{L}$ is not a subspace of $\mathbb{R}^2$.

20 (a) You need to know that a sum of two continuous functions is continuous, as is a scalar multiple of a continuous function (see Adams, Section 1.4, Theorem 6).
(b) Suppose $f(a) = f(b)$ and $g(a) = g(b)$. Then, clearly, $(f+g)(a) = (f+g)(b)$. Also $(cf)(a) = (cf)(b)$, so the set of functions under consideration is closed under addition and scalar multiplication, hence a subspace of $C[a,b]$.

24. True, True, True (of itself),
False, though it is *isomorphic* to a subspace of $\mathbb{R}^3$, for example the subspace of all vectors whose z-component is zero,
False, since it doesn't say what $\boldsymbol{u}$ and $\boldsymbol{v}$ are. The statement would be true if it read instead : (ii) for any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, it is also the case that $\boldsymbol{u}+\boldsymbol{v}$ is in H (iii) if $\boldsymbol{u}$ is in H then so is $c\boldsymbol{u}$, for any scalar c.

36. $\boldsymbol{y}$ is in Col(A) if and only if the system $A\boldsymbol{x} = \boldsymbol{y}$ has a solution. So run the command "$> A\backslash\boldsymbol{y}$" and see if you get an error message. Alternatively run "rref([A $\boldsymbol{y}$])". It turns out there is a solution, $\boldsymbol{x} = \frac{1}{5}[-1 \ -2 \ 3]^T$. In other words,

$$\boldsymbol{y} = -\frac{1}{5}\begin{bmatrix} 3 \\ 8 \\ -5 \\ 2 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -5 \\ 7 \\ -8 \\ -2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} -9 \\ -6 \\ 3 \\ -9 \end{bmatrix}.$$

4.2

26. True, True, False, True, True, True (don't bother yet as to why).
30. Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be any two vectors in the range of T and c any scalar. We must show that both $\boldsymbol{w}_1+\boldsymbol{w}_2$ and $c\boldsymbol{w}_1$ are in the range of T.

Since both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are in the range of T there exist, by definition, vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in V such that

$$T(\boldsymbol{v}_1) = \boldsymbol{w}_1, \qquad T(\boldsymbol{v}_2) = \boldsymbol{w}_2.$$

But T is linear, thus

$$T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2) = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

and

$$T(c\boldsymbol{v}_1) = cT(\boldsymbol{v}_1) = c\boldsymbol{w}_1.$$

This equations show that both $\boldsymbol{w}_1 + \boldsymbol{w}_2$ and $c\boldsymbol{w}_1$ are in the range of T, as desired.

4.3

4. The matrix with these three vectors as its columns can be Gauss-reduced, via the row operations

$$R_1 \mapsto \frac{1}{2}R_1, \quad R_2 \mapsto R_2 + R_1, \quad R_3 \mapsto R_3 - R_1, \quad R_3 \mapsto 2R_3 + R_2,$$

to the echelon form

$$\begin{bmatrix} 1 & 1 & -4 \\ 0 & -2 & 1 \\ 0 & 0 & 17 \end{bmatrix}.$$

Since we have a pivot in each column, the three vectors are linearly independent and form a basis for $\mathbb{R}^3$.

10. The variables x_4 and x_5 are free. Back substitution can be verified to yield

$$x_3 = 2x_5, \quad x_2 = x_4 + 2x_5, \quad x_1 = -2x_4 - 3x_5,$$

from which we can further calculate that $\{[-2\ 1\ 0\ 1\ 0]^T, [-3\ 2\ 2\ 0\ 1]^T\}$ is a basis for the nullspace.

22. False, True, True, False, False (rather the corresponding columns in A itself).

30. Let A be the $n \times k$ matrix which has these vectors as its columns.

If these vectors formed a basis for $\mathbb{R}^n$ then, in particular, they would be linearly independent. This would mean that Nul(A) would contain only the zero vector. But since A has more columns than rows, there will remain at least one free variable after Gauss elimination on A, and thus Nul(A) contains non-zero vectors (see Section 1.5, for example, though I don't know what theorem exactly he wants you to use (and it doesn't matter !)).

36. Let A be the matrix with $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ as its rows, let B be the matrix with $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ as its rows, and let C be the 6-row matrix got by adjoining B below A - in MATLAB one would write "$C = [A; B]$". Now compute RREF for each matrix. One obtains

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 & 16/3 \\ 0 & 1 & 0 & -32/3 \\ 0 & 0 & 1 & 23/3 \end{bmatrix},$$

$$\mathrm{RREF}(C) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The non-zero rows in these three matrices form bases for, respectively, H, K and $H + K$. In particular, $H + K = \mathbb{R}^4$, whereas $\dim(H) = 2$ and $\dim(K) = 3$.

38. Actually, you don't need MATLAB to solve this exercise. If we can find seven numbers $t_1, ..., t_7$ such that the values $a_i := \cos t_i$ are all distinct, then we'll have a 7×7 system $A\boldsymbol{x} = \mathbf{0}$, where $\boldsymbol{x} = [c_0 \ c_1 \cdots c_6]^T$ and A is a Vandermonde matrix. Since a Vandermonde matrix is known to be invertible (see Supplementary Exercise 11(b) in Chapter 2, plus a bunch of other exercises in the book on Vandermonde matrices (check index at back))), this will imply that the system has only the trivial solution, which is what we need to show. The point is, it's pretty obvious one can find seven (indeed, as many as you want) values of t for which the corresponding values of $\cos(t)$ are all distinct, since the range of the function $\cos(t)$ is the whole interval $[-1, 1]$.

4.4

10. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$.

16. True, False (other way round), True (namely, if the plane passes through the origin).

4.5

6. Write out the subspace more explicitly as

$$\left\{ a \cdot \begin{bmatrix} 3 \\ 0 \\ -7 \\ -3 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ -1 \\ 6 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} -1 \\ -3 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

The dimension of the subspace thus equals the rank of the matrix

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & -3 \\ -7 & 6 & 5 \\ -3 & 0 & 1 \end{bmatrix}.$$

The row operations

$$R_3 \mapsto 3R_3 + 7R_1, \quad R_4 \mapsto R_4 + R_1, \quad R_3 \mapsto R_3 + 18R_2,$$

reduce the matrix to the echelon form

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & -46 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the rank is 3.

14. The column space has dimension 4 (columns 1,4,5,7 form a basis), whereas the nullspace has dimension 3 (variables x_2, x_3, x_6 are free).

20. False (see 4.1.24(d)), False (rather the number of *free* variables), False, False (see 19(d)), True.

30. False, True, False.

4.6

18. False (see 4.3.22(e)), False, True, True, True.
30. They must be equal, since consistency means that $\boldsymbol{b}$ is a linear combination of the columns of A, hence adding $\boldsymbol{b}$ as a column to the matrix does not increase the dimension of its column space.

4.7

10. We have that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [\boldsymbol{c}_1\ \boldsymbol{c}_2]^{-1}[\boldsymbol{b}_1\ \boldsymbol{b}_2] = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 4 \\ -12 & 2 \end{bmatrix} =$$

$$= \frac{1}{30}\begin{bmatrix} 9 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -12 & 2 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 90 & 30 \\ -60 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}.$$

On the other hand,

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}.$$

12. True, False (rather it satisfies $[\boldsymbol{x}]_{\mathcal{C}} = P[\boldsymbol{x}]_{\mathcal{B}}$).

5.1

6. Compute

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix}.$$

The right-hand side is not a multiple of $[1\ -2\ 2]^T$, hence the latter is not an eigenvector of the matrix.
22. False (it's true if $\boldsymbol{x}$ is not the zero vector), False (opposite true), True, False, True.

5.2

18. We have

$$A - 4I_4 = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

The row operations $R_2 \mapsto R_2 + R_1$ and $R_4 \mapsto 7R_4 + R_3$ produce the echelon form

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the eigenspace to be 2-dimensional, we require that the nullspace of this matrix be 2-dimensional. This happens if and only if $h + 3 = 0$, i.e.: if and only if $h = -3$.
20. We know that for any matrix M it holds that $\det M = \det M^T$. Let λ be a scalar. Then

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I).$$

Thus $\det(A - \lambda I) = 0$ if and only if $\det(A^T - \lambda I) = 0$. In other words, λ is an eigenvalue of A if and only if it is an eigenvalue of A^T, v.s.v.
22. False (the volume equals $|\det A|$), False, True,

False : as an example, take $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This is diagonal, so its only eigenvalue is $\lambda = 1$. The row replacement $R_2 \mapsto R_2 - R_1$ produces the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. One can check that the characteristic polynomial for this matrix is $\lambda^2 - \lambda + 1$, so there are two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{3}i\right)$.

24. Similarity means that there exists an invertible matrix P such that $B = P^{-1}AP$. Then

$$\det B = \det(P^{-1}AP) = (\det P^{-1})(\det A)(\det P) = \left(\frac{1}{\det P}\right)(\det A)(\det P) = \det A, \quad \text{v.s.v.}$$

30. The command "poly(A)" produces the coefficients of the characteristic polynomial of a matrix A, whereas the command "eig(A)" produces its eigenvalues. Using short format, I obtained the following table of values :

a	$p(t)$	$\lambda_1, \lambda_2, \lambda_3$
32	$t^3 - 4t^2 + 5t - 2$	$2, 1, 1$
31.9	$t^3 - 4t^2 + 3.8t - 0.8$	$2.7042, 0.2958, 1$
31.8	$t^3 - 4t^2 + 2.6t + 0.4$	$3.1279, -0.1279, 1$
32.1	$t^3 - 4t^2 + 6.2t - 3.2$	$1.5 \pm 0.9747i, 1$
32.2	$t^3 - 4t^2 + 7.4t - 4.4$	$1.5 \pm 1.4663i, 1$

A couple of things are noteworthy :

(i) 1 is always an eigenvalue, because in the matrix $A - I_3$, the second row is $-\frac{4}{7}$ of the first row.

(ii) We can see that if we replace a by $a + 0.1$, then the coefficients of t^0 and t^1 in the characteristic polynomial change by -1.2 and $+1.2$ respectively. Hence, one can verify that, for arbitrary a, we will have $p(t) = t^3 - 4t^2 + (-379 + 12a)t + (382 - 12a)$.

5.3

22. False (true if we add the words 'linearly independent'), False (converse true), True, False.

28. That the $n \times n$ matrix A has n linearly independent vectors means that it is diagonalisable, i.e.: $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. But then, by Theorems 2.1.3(d) and 2.2.6(c), we have

$$A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D P^T,$$

where we have also used the fact that $D^T = D$, since D is diagonal. Let $Q := P^T$. We have shown that $A^T = QDQ^{-1}$. Since Q is invertible, we've shown that A^T is also diagonalisable, hence it also has n linearly independent eigenvectors (namely the columns of Q, which are the rows of P).

5.4

6. We have

$$T(\boldsymbol{p}(t)) = \boldsymbol{p}(t) + 2t^2\boldsymbol{p}(t) = (1 + 2t^2)\boldsymbol{p}(t). \tag{6}$$

(a) From (6) it follows that

$$T(3 - 2t + t^2) = (1 + 2t^2)(3 - 2t + t^2) = 3 - 2t + 7t^2 - 4t^3 + 2t^4.$$

(b) It's required to show, for all polynomials $\boldsymbol{p_1}(t)$, $\boldsymbol{p_2}(t) \in \mathbb{P}_2$ and for all scalars $c_1, c_2 \in \mathbb{R}$, that

$$T(c_1\boldsymbol{p_1}(t) + c_2\boldsymbol{p_2}(t)) = c_1T(\boldsymbol{p_1}(t)) + c_2T(\boldsymbol{p_2}(t)). \tag{7}$$

Using (6), we can verify (7) as follows:

$$\begin{aligned}\text{VL} &= T(c_1\boldsymbol{p_1}(t) + c_2\boldsymbol{p_2}(t)) \\ &= (1 + 2t^2)(c_1\boldsymbol{p_1}(t) + c_2\boldsymbol{p_2}(t)) \\ &= c_1\left[(1 + 2t^2)\boldsymbol{p_1}(t)\right] + c_2\left[(1 + 2t^2)\boldsymbol{p_2}(t)\right] \\ &= c_1T(\boldsymbol{p_1}(t)) + c_2T(\boldsymbol{p_2}(t)) = \text{HL, v.s.v.}\end{aligned}$$

(c) Let $\mathcal{B} := \{1, t, t^2\}$ and $\mathcal{C} := \{1, t, t^2, t^3, t^4\}$, and let $[T]_{\mathcal{B},\mathcal{C}}$ denote the matrix for T w.r.t. these bases for $\mathbb{P}_2$ and $\mathbb{P}_4$ respectively. Then, by definition,

$$[T]_{\mathcal{B},\mathcal{C}} = \left([T(1)]_{\mathcal{C}}\ [T(t)]_{\mathcal{C}}\ [T(t^2)]_{\mathcal{C}}\right), \tag{8}$$

in other words, the columns of the matrix are obtained by evaluating T on each vector in the basis $\mathcal{B}$ and writing the results in terms of the basis $\mathcal{C}$.

By (6), we have

$$\begin{aligned}T(1) &= 1 + 2t^2 = (1\ 0\ 2\ 0\ 0)^T, \\ T(t) &= t + 2t^3 = (0\ 1\ 0\ 2\ 0)^T, \\ T(t^2) &= t^2 + 2t^4 = (0\ 0\ 1\ 0\ 2)^T.\end{aligned}$$

Substituting into (8) we get

$$[T]_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

12. It follows from what we did in lectures about base-change for linear transformations that

$$[T]_{\mathcal{B}} = P^{-1}AP,$$

where P is the matrix whose columns are the vectors in the basis $\mathcal{B}$, i.e.: $P = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Hence, in the present exercise,

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -6 & -2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \cdots = \begin{bmatrix} -4 & 0 \\ 2 & -2 \end{bmatrix}.$$

Note that, what this means explicitly is that

$$A\boldsymbol{b}_1 = T(\boldsymbol{b}_1) = -4\boldsymbol{b}_1 + 2\boldsymbol{b}_2 \ \text{ and } \ A\boldsymbol{b}_2 = T(\boldsymbol{b}_2) = 0 \cdot \boldsymbol{b}_1 - 2\boldsymbol{b}_2 = -2\boldsymbol{b}_2.$$

One can multiply out and check that these equations hold.

20. If A is similar to B it means that there is some invertible matrix X such that $B = X^{-1}AX$. Now square both sides of this equation, and we get

$$\begin{aligned} B^2 = (X^{-1}AX)^2 &= (X^{-1}AX)(X^{-1}AX) = (X^{-1}A)(XX^{-1})(AX) \\ &= (X^{-1}A)I_n(AX) = X^{-1}AAX = X^{-1}A^2X. \end{aligned}$$

In other words, there is some invertible matrix, namely the same matrix X, such that $B^2 = X^{-1}A^2X$. This, by definition of similarity, means that A^2 is similar to B^2, v.s.v.

32. The diagonaising basis consists of eigenvectors of A. These are found by running the command "$[V, D] = eig(A)$". When I did this in short format I got

$$V = \begin{bmatrix} -0.6325 & 0.2626 & -0.5968 & 0.6614 \\ -0.3162 & -0.1313 & -0.5613 & -0.5203 \\ 0.3162 & -0.9191 & -0.5731 & -0.1264 \\ -0.6325 & 0.2626 & -0.0158 & 0.5252 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

The columns of V form the desired basis. Note that it's pretty clear that we could choose simpler eigenvectors in the first and second columns, namely $[-2\ \ -1\ 1\ \ -2]^T$ and $[2\ \ -1\ \ -7\ 2]^T$ respectively, though there are round-off errors in the first column. One can check directly that these integer-valued alternatives are still eigenvectors for $\lambda_1 = 5$ and $\lambda_2 = 1$ respectively. No such simplification seems to be possible with the third and fourth columns of V, though we can drop all the minus signs in the third column.

5.7

6. The solution is

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{bmatrix} = \begin{bmatrix} 5e^{-t} - 2e^{-2t} \\ 5e^{-t} - 3e^{-2t} \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}.$$

The origin is an attractor and the direction of greatest attraction is along the line $2y = 3x$.

6.1

6. $\frac{5}{49}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$.

8. 7.

20. True, false (rather $|c|$), true, true, true.

24. We have

$$||\boldsymbol{u} \pm \boldsymbol{v}||^2 = (\boldsymbol{u} \pm \boldsymbol{v}) \cdot (\boldsymbol{u} \pm \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} \pm 2(\boldsymbol{u} \cdot \boldsymbol{v}) = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 \pm 2(\boldsymbol{u} \cdot \boldsymbol{v}).$$

When we add, the terms $\pm 2(\boldsymbol{u} \cdot \boldsymbol{v})$ cancel and we're left with the right-hand side.

26. W is the nullspace of the 1×3 matrix with the single row $\boldsymbol{u}^T$. So he's probably referring to some theorem that says that the nullspace of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. Geometrically, W is a plane through the origin with normal vector $\boldsymbol{u}$. Its equation is $5x - 6y + 7z = 0$.

28. The hypothesis is that $\boldsymbol{y} \cdot \boldsymbol{u} = \boldsymbol{y} \cdot \boldsymbol{v} = 0$. Now let $\boldsymbol{w} \in \text{Span}\{\boldsymbol{u}, \boldsymbol{v}\}$. Then there exist real numbers c_1, c_2 such that $\boldsymbol{w} = c_1\boldsymbol{u} + c_2\boldsymbol{v}$. But then, by Theorem 6.1.1(b,c), we have

$$\boldsymbol{y} \cdot \boldsymbol{w} = \boldsymbol{y} \cdot (c_1\boldsymbol{u} + c_2\boldsymbol{v}) = c_1(\boldsymbol{y} \cdot \boldsymbol{u}) + c_2(\boldsymbol{y} \cdot \boldsymbol{v}) = c_1 \cdot 0 + c_2 \cdot 0 = 0, \text{ v.s.v.}$$

30. Well, just follow the steps outlined, and employ Theorem 6.1.1.

34. If we compute rref(A), then the non-zero columns form a basis for the rowspace. Using rational format, this gave me

$$R = \begin{bmatrix} 1 & 0 & 5 & 0 & -1/3 \\ 0 & 1 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix}.$$

The command "null(A)" produces an (orthonormal) basis for the nullspace. Using short format, this gave me

$$N = \begin{bmatrix} -0.8120 & -0.5340 \\ 0.3504 & -0.6888 \\ 0.1894 & 0.0762 \\ -0.1349 & 0.1532 \\ 0.4048 & -0.4595 \end{bmatrix}.$$

By Theorem 6.1.3, we should have $RN = O_{3\times 2}$. In fact, due to roundoff errors, here is what Matlab returned, in short format :

$$RN = 10^{-15} \cdot \begin{bmatrix} 0.2776 & -0.8327 \\ 0.1110 & -0.2220 \\ -0.0555 & 0 \end{bmatrix}.$$

6.2

24. False, false (true if $||\boldsymbol{u}_i|| = 1$ for each $i = 1, ..., p$), true, true, true.

6.3

22. True, true, true, false (rather $\text{proj}_W\, \boldsymbol{y}$), false (true when $n = p$).

6.4

18. True, true, true.
22. I don't want to write out all the details, but I'll give you the idea. What you need to show is that, for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ and any $c_1, c_2 \in \mathbb{R}$, one has

$$\text{proj}_W(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) = c_1 \left(\text{proj}_W\, \boldsymbol{x}_1\right) + c_2 \left(\text{proj}_W\, \boldsymbol{x}_2\right). \tag{9}$$

To verify (9), employ Theorems 6.1.1 and 6.3.8.

6.5

18. True, false, true, false (true if A^TA is invertible), ?? (don't understand what he means by 'reliable'), True (I guess).

6.6

4. $y = \frac{1}{10}(43 - 7x)$.
10 (a) The model is $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} e^{-.02t_1} & e^{-.07t_1} \\ e^{-.02t_2} & e^{-.07t_2} \\ e^{-.02t_3} & e^{-.07t_3} \\ e^{-.02t_4} & e^{-.07t_4} \\ e^{-.02t_5} & e^{-.07t_5} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} M_A \\ M_B \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}.$$

15, 16. In matrix form, we're being asked to compute

$$\hat{\boldsymbol{x}} = (A^T A)^{-1}(A^T \boldsymbol{b}), \tag{10}$$

where

$$A = \begin{bmatrix} 1 & x_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_n \end{bmatrix}, \quad \hat{\boldsymbol{x}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}.$$

One may verify that the so-called *normal equations* are obtained by instead writing (10) in the form $(A^T A)\hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ and multiplying out the matrices. If we invert $A^T A$, and instead solve (10) explicitly, it turns out we get

$$\hat{\beta}_0 = \frac{(\sum x^2)(\sum y) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}, \quad \hat{\beta}_1 = \frac{-(\sum x)(\sum y) + n(\sum xy)}{n(\sum x^2) - (\sum x)^2}.$$

7.1

2. Not symmetric.
4. Symmetric.
6. Not symmetric.
8. The matrix is orthogonal and

$$A^{-1} = A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

10. The matrix is not orthogonal since, though its columns are pairwise orthogonal vectors, they are not unit vectors - each has length 3. Note that this implies that $\frac{1}{3}A$ is an orthogonal matrix, which is also symmetric. Hence, we can still easily write down A^{-1}, since

$$A^{-1} = \left(3 \cdot \frac{1}{3}A\right)^{-1} = 3^{-1} \cdot \left(\frac{1}{3}A\right)^{-1} = \frac{1}{3} \cdot \left(\frac{1}{3}A\right)^{T} = \frac{1}{3} \cdot \left(\frac{1}{3}A\right) = \frac{1}{9}A = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

24. Check directly that $A\boldsymbol{v}_1 = 10\boldsymbol{v}_1$ and $A\boldsymbol{v}_2 = \boldsymbol{v}_2$. Thus we have at least two eigenvalues, $\lambda_1 = 10$ and $\lambda_2 = 1$. The matrix A is symmetric, so we know it must be orthogonally diagonalisable. Therefore, there must be a third eigenvector $\boldsymbol{v}_3$, which is orthogonal to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. But, since we're working in $\mathbb{R}^3$, there is only one possibility for such a vector, up to a scalar multiple, namely we can take

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} - 4\vec{k}.$$

Thus $\boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ must be an eigenvector. Now check directly that $A\boldsymbol{v}_3 = \boldsymbol{v}_3$. Thus the eigenvalue here is also $\lambda_2 = 1$, so this eigenspace is two-dimensional.

We then have an orthogonal diagonalisation

$$A = PDP^T,$$

where

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} | & | & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ | & | & | \end{bmatrix}$$

and $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are the normalised eigenvectors, i.e.:

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||} = \frac{\boldsymbol{v}_1}{3}, \quad \boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{||\boldsymbol{v}_2||} = \frac{\boldsymbol{v}_2}{\sqrt{2}}, \quad \boldsymbol{u}_3 = \frac{\boldsymbol{v}_3}{||\boldsymbol{v}_3||} = \frac{\boldsymbol{v}_3}{\sqrt{18}}.$$

Thus finally

$$P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & -4/\sqrt{18} \end{bmatrix}.$$

26. True, true, false, true.
28.

$$(A\boldsymbol{x}) \cdot \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y} = (\boldsymbol{x}^T A^T)\boldsymbol{y} \overset{A=A^T}{=} (\boldsymbol{x}^T A)\boldsymbol{y} = \boldsymbol{x}^T(A\boldsymbol{y}) = \boldsymbol{x} \cdot (A\boldsymbol{y}), \quad \text{v.s.v.}$$

7.2

22. True, False (P must be a matrix which orthogonally diagonalises A), False (without some extra conditions on A and c), False, True.