Eigenvalues of matrices

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TMV142/186
13 feb 2017

A Predator-prey model

	seals (thousands)	cod (tons)
initial	s_0	c_0

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year 2	$s_2 = 0.6s_1 + 0.5c_1$	$c_2 = -0.24s_1 + 1.4c_1$
$\vdots$	$\vdots$	$\vdots$
year k	$s_k = 0.6s_{k-1} + 0.5c_{k-1}$	$c_k = -0.24s_{k-1} + 1.4c_{k-1}$
$\vdots$	$\vdots$	$\vdots$

Matrix formulation

Population vectors:

$$\begin{bmatrix} s_k \\ c_k \end{bmatrix}$$

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Difference equation:

$$\begin{bmatrix} s_k \\ c_k \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ -0.24 & 1.4 \end{bmatrix} \begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix}$$

A key question

What is the long-term behaviour of the model?

Stable populations

$$\begin{bmatrix} s_k \\ c_k \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ -0.24 & 1.4 \end{bmatrix} \begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix}$$

Stable populations

$$\begin{bmatrix} s_k \\ c_k \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ -0.24 & 1.4 \end{bmatrix} \begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix} = \lambda \begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix}$$

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λ is called an eigenvalue of the matrix

$$\begin{bmatrix} 0.6 & 0.5 \\ -0.24 & 1.4 \end{bmatrix}$$

if the equation has a solution

$$\begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Such a vector is called an eigenvector of the matrix.

Computing the eigenvalues

Consider

$$\begin{bmatrix} 0.6 & 0.5 \\ -0.24 & 1.4 \end{bmatrix} \begin{bmatrix} s \\ c \end{bmatrix} = \lambda \begin{bmatrix} s \\ c \end{bmatrix}$$

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Performing the matrix multiplication and rearranging terms, we get

$$(0.6 - \lambda)s + 0.5c = 0 \quad (1)$$

$$-0.24s + (1.4 - \lambda)c = 0 \quad (2)$$

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Substituting this into Eq. (2), we obtain

$$-0.24s - \frac{(1.4-\lambda)(0.6-\lambda)}{0.5}s = 0$$

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Substituting this into Eq. (2), we obtain

$$-0.24s - \frac{(1.4 - \lambda)(0.6 - \lambda)}{0.5}s = 0$$

or equivalently

$$\left[0.12 + (1.4 - \lambda)(0.6 - \lambda)\right]s = 0$$

Computing the eigenvalues

Solving

$$0.12 + (1.4 - \lambda)(0.6 - \lambda) = 0$$

we find the two eigenvalues

$$\lambda_1 = 1.2$$
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Corresponding eigenvectors are given by

$$p_1 = \begin{bmatrix} 300 \\ 360 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 500 \\ 200 \end{bmatrix}$$

Long-term behaviour

An example:

$$\begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} 500 \\ 300 \end{bmatrix} = \frac{5}{12} \begin{bmatrix} 300 \\ 360 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 500 \\ 200 \end{bmatrix}$$

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$$\begin{bmatrix} s_k \\ c_k \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ -0.24 & 1.4 \end{bmatrix}\begin{bmatrix} s_{k-1} \\ c_{k-1} \end{bmatrix}$$

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Long-term behaviour