RÖ 3. PARTIELL INTEGRATION, PARTIALBRÅKSUPPDELNING

Partiell integration

"Behåll den integrerade och derivera den andra"

$$\int_a^b f(x)g(x)\,dx = \underbrace{\left[f(x)G(x)\right]_a^b}_{\text{"Behåll den ena och integrera den andra"}} - \int_a^b f'(x)G(x)\,dx$$

G – primitiv funktion

- Varför fungerar detta?
- $$\int_a^b f(x)g(x)\,dx + \int_a^b f'(x)G(x)\,dx = \left[f(x)G(x)\right]_a^b$$

$$\Leftrightarrow$$

$$\int_a^b \frac{d}{dx}\left(f(x)G(x)\right)dx = \left[f(x)G(x)\right]_a^b$$

Integralkalkylens fundamentalsats!

- Partialbråksuppdelning

Exempel: $\displaystyle\int \frac{x^2+2x}{(x+1)(2x+3)^2(x^2-1)^2}\,dx$

Vill skriva

$$\frac{x^2+2x}{(x+1)(2x+3)^2(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{2x+3} + \frac{C}{(2x+3)^2}$$

$$+ \frac{Dx+E}{x^2-1} + \frac{Fx+G}{(x^2-1)^2}$$ och lösa ut A, B, C, D, E, F, G.

Då kan vi integrera varje term för sig sedan

A.6.1.6. Beräkna $\int x(\ln x)^3 dx$.

Derivera $f(x) = (\ln x)^3$ och integrera x så kommer vi så småningom till något enklare!

$$\int x(\ln x)^3 dx = \frac{x^2}{2}(\ln x)^3 - \frac{1}{2}\int x^2 \cdot 3(\ln x)^2 \cdot \frac{1}{x} dx$$

$$= \frac{x^2}{2}(\ln x)^3 - \frac{3}{2}\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^3 - \frac{3}{2}\left[\frac{x^2}{2}(\ln x)^2 - \int \frac{x^2}{2} 2\ln x \cdot \frac{1}{x} dx\right]$$

$$= \frac{x^2}{2}(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2}\int x \ln x \, dx$$

$$= \frac{x^2}{2}(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2}\left[\frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx\right]$$

$$= \frac{x^2}{2}(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{4}x^2 \ln x - \frac{3}{4} \cdot \frac{x^2}{2} + C$$

$$= \frac{x^2}{8}\left(4(\ln x)^3 - 6(\ln x)^2 + 6\ln x - 3\right) + C$$

A.6.1.10. Beräkna $\int x^5 e^{-x^2} dx$

Här vill vi "derivera bort" x^5, så vi får integrera e^{-x^2}.

Sätt $t = -x^2 \Rightarrow \frac{dt}{dx} = -2x \Rightarrow dt = -2x\,dx$

$$\int x^5 e^{-x^2} dx = -\int x^4 e^{-x^2} \frac{1}{2} \underbrace{-2x\,dx}_{dt} = \frac{-1}{2}\int t^2 e^t dt$$

$$= -\frac{1}{2}\left[t^2 e^t - \int 2te^t dt\right] = -\frac{1}{2}t^2 e^t + \int te^t dt = -\frac{1}{2}t^2 e^t + \longrightarrow$$

$$+te^t - \int e^t dt = -\frac{1}{2}t^2e^t + te^t - e^t + C = \frac{e^t}{2}(-t^2+2t-2) + C$$

$$= \{t = -x^2\} = \frac{e^{-x^2}}{2}(-x^4 - 2x^2 - 2) + C$$

A.6.1 32 Bestäm en reduktionsformel för $I_n = \int_0^{\pi/2} x^n \sin x \, dx$.
och räkna ut I_6.

Vi vill beskriva I_n som en funktion av $I_{n-1}, I_{n-2}, \ldots, I_0$.

- Idé: Derivera x^n och integrera $\sin x$ så kommer vi så småningom tillbaka till $x^p \cdot \sin x$, $p<n$.

- $$I_n = \int_0^{\pi/2} x^n \sin x \, dx = \left[-x^n \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} -nx^{n-1}\cos x \, dx = 0 + n\int_0^{\pi/2} x^{n-1}\cos x \, dx$$

$$= n\left[x^{n-1}\sin x\right]_0^{\pi/2} - n\int_0^{\pi/2}(n-1)x^{n-2}\sin x \, dx = n\left(\frac{\pi}{2}\right)^{n-1}\cdot 1 - n(n-1)I_{n-2}, \; n \geq 2$$

- $$I_6 = 6\cdot\left(\frac{\pi}{2}\right)^5 - 6\cdot 5\cdot I_4 = 6\cdot\frac{\pi^5}{32} - 30\left[4\left(\frac{\pi}{2}\right)^3 - 4\cdot 3\cdot I_2\right]$$

$$= \frac{3}{16}\pi^5 - 15\pi^3 + 360 I_2 = \frac{3}{16}\pi^5 - 15\pi^3 + 360\left[2\cdot\frac{\pi}{2} - 2\cdot 1\cdot I_0\right]$$

$$= \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720 I_0$$

$$I_0 = \int_0^{\pi/2} x^0 \sin x \, dx = \int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2} = 0 - (-1) = 1$$

$$\Rightarrow I_6 = \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720$$

.2.18. Beräkna $\int \frac{1}{x^4-a^4}\,dx$

$$\int \frac{1}{x^4-a^4}\,dx = \int \frac{1}{(x^2)^2-(a^2)^2}\,dx = \int \frac{1}{(x^2-a^2)(x^2+a^2)}\,dx$$

Ansätt $\frac{1}{(x^2-a^2)(x^2+a^2)} = \frac{Ax+B}{x^2-a^2} + \frac{Cx+D}{x^2+a^2} \iff$

$$1 = (Ax+B)(x^2+a^2) + (Cx+D)(x^2-a^2) \iff$$

$$1 = Ax^3 + Aa^2x + Bx^2 + Ba^2 + Cx^3 - Ca^2x + Dx^2 - Da^2 \Rightarrow$$

x^3 : $A + C = 0 \Rightarrow A = -C$

x^2 : $B + D = 0 \Rightarrow B = -D$

x : $(A-C)a^2 = 0 \Rightarrow 2A = 0 \Rightarrow A = C = 0$

1 : $(B-D)a^2 = 1 \Rightarrow 2Ba^2 = 1 \Rightarrow B = \frac{1}{2a^2} \Rightarrow D = -\frac{1}{2a^2}$

Vi får $\int \frac{1}{x^4-a^4}\,dx = \frac{1}{2a^2}\int \frac{1}{x^2-a^2}\,dx - \frac{1}{2a^2}\int \frac{1}{x^2+a^2}\,dx$

$$= \frac{1}{2a^2}\cdot\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^2}\cdot\frac{1}{a}\arctan\frac{x}{a} + C$$

$$= \frac{1}{4a^3}\left(\ln\left|\frac{x-a}{x+a}\right| - 2\arctan\frac{x}{a}\right) + C$$

A.6.2.23 Beräkna $\int \frac{1}{(x^2-1)^2}\,dx$

Notera att $\frac{1}{(x^2-1)^2} = \frac{1}{(x+1)^2(x-1)^2}$ och partialbråksuppdela.

$$\frac{1}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{A(x+1)(x-1)^2 + B(x-1)^2 + C(x+1)^2(x-1) + D(x+1)^2}{(x+1)^2(x-1)^2}$$

$$\iff A(x+1)(x-1)^2 + B(x-1)^2 + C(x+1)^2(x-1) + D(x+1)^2 = 1$$

$$\iff A(x+1)(x^2-2x+1) + B(x^2-2x+1) + C(x^2+2x+1)(x-1) + D(x^2+2x+1) = 1$$

$$\iff A(x^3-2x^2+x+x^2-2x+1) + B(x^2-2x+1) + C(x^3-x^2+2x^2-2x+x-1) + D(x^2+2x+1) = 1$$

$\Rightarrow$ x^3: $A + C = 0$ $\Rightarrow$ $A = -C$

x^2: $-A + B + C + D = 0$ $\Rightarrow$ $2C + B + D = 0$ $\Rightarrow$ $C = -\frac{1}{2}(B+D)$

x: $-A - 2B - C + 2D = 0$ $\Rightarrow$ $B = D$ (eftersom $A + C = 0$)

1: $A + B - C + D = 1$ $\Rightarrow$ $\frac{1}{2}(2B) + B + \frac{1}{2}(2B) + B = 1$ $\Rightarrow$ $B = \frac{1}{4}$

$\Rightarrow$ $D = \frac{1}{4}$ $\Rightarrow$ $C = -\frac{1}{4}$ $\Rightarrow$ $A = \frac{1}{4}$

Vi får:

$$\int \frac{1}{(x^2-1)^2}\,dx = \int \frac{1}{(x+1)^2(x-1)^2}\,dx = \frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{(x+1)^2}\,dx$$

$$-\frac{1}{4}\int \frac{1}{x-1}\,dx + \frac{1}{4}\int \frac{1}{(x-1)^2}\,dx = \frac{1}{4}\ln|x+1| - \frac{1}{4}\frac{1}{(x+1)} - \frac{1}{4}\ln|x-1|$$

$$-\frac{1}{4}\frac{1}{(x-1)} + C = \frac{1}{4}\left(\ln\left|\frac{x+1}{x-1}\right| - \frac{2x}{x^2-1}\right) + C$$