RÖ 4. INVERSSUBSTITUTION, GENERALISERADE INTEGRALER

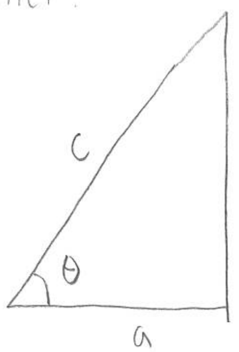
Inverssubstitution

Knep för att lösa integraler som innehåller

① $\sqrt{a^2+b^2}$

② $\sqrt{c^2-a^2}$

③ $\sqrt{c^2-b^2}$



$c=\sqrt{a^2+b^2}$

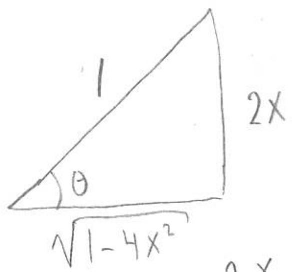
$b=\sqrt{c^2-a^2}$

$a=\sqrt{c^2-b^2}$

- Substituera $\tan\theta = \frac{b}{a}$, $\sin\theta = \frac{b}{c}$ eller $\cos\theta = \frac{a}{c}$, där en av
- a, b, c är en linjär funktion av x i ①, ②, ③, t.ex.

$$\sqrt{2x^2-4}\ , \quad c=\sqrt{2}\,x\ ,\quad a=2 \text{ eller } b=2.$$

A. 6.3.4 Beräkna $\int \frac{1}{x\sqrt{1-4x^2}}\,dx$



- Vi ser att $\cos\theta = \sqrt{1-4x^2}$
  och $\sin\theta = 2x \iff x = \frac{1}{2}\sin\theta$, $\tan\theta = \frac{2x}{\sqrt{1-4x^2}}$
- Vi får enklast dθ om vi deriverar uttrycket som innehåller $\sin\theta$.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\sin\theta\right) = \frac{1}{2}\cos\theta \Rightarrow dx = \frac{1}{2}\cos\theta\, d\theta \ ,\ \text{så}$$

$$\int \frac{1}{x\sqrt{1-4x^2}}\,dx = \int \frac{1}{\frac{1}{2}\sin\theta\cos\theta}\cdot\frac{1}{2}\cos\theta\, d\theta = \int \frac{1}{\sin\theta}\,d\theta$$

$$= -\ln\left|\frac{1}{\sin\theta}+\frac{1}{\tan\theta}\right| + C = -\ln\left|\frac{1}{2x}+\frac{\sqrt{1-4x^2}}{2x}\right| + C =$$

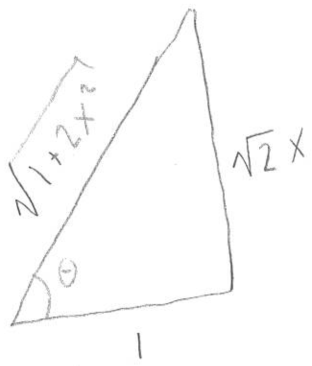
$\ln\left|\left(\frac{1+\sqrt{1-4x^2}}{x}\right)/2\right| + C = \left\{ \ln\frac{a}{b} = \ln a - \ln b \right\}$

$-\ln\left|\frac{1+\sqrt{1-4x^2}}{x}\right| + \underbrace{\ln 2 + C}_{C'} = -\ln\left|\frac{1+\sqrt{1-4x^2}}{x}\right| + C'$

lan kan integrera $\frac{1}{\sin\theta}$ $\left(\int \frac{1}{\sin\theta}\,d\theta\right)$ genom att

ubstituera $t = \frac{1}{\sin\theta} + \frac{1}{\tan\theta}$

.3.14. Beräkna $\int \frac{1}{(1+2x^2)^{5/2}}\,dx$



Vi har $\sin\theta = \frac{\sqrt{2}\,x}{\sqrt{1+2x^2}}$,

$\cos\theta = \frac{1}{\sqrt{1+2x^2}}$, $\tan\theta = \sqrt{2}x \iff x = \frac{1}{\sqrt{2}}\tan\theta$

Vi får enklast $d\theta$ genom att derivera uttrycket för $\tan\theta$.

$\frac{dx}{d\theta} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\cos^2\theta} \Rightarrow dx = \frac{1}{\sqrt{2}\cos^2\theta}\,d\theta$, så

$\int \frac{1}{(1+2x^2)^{5/2}}\,dx = \int \frac{1}{(\sqrt{1+2x^2})^5}\,dx = \int \cos^5\theta \cdot \frac{1}{\sqrt{2}\cos^2\theta}\,d\theta$

$= \frac{1}{\sqrt{2}}\int \cos^3\theta\,d\theta = \left\{\cos^2\theta = 1-\sin^2\theta\right\} = \frac{1}{\sqrt{2}}\int (1-\sin^2\theta)\cos\theta\,d\theta =$

$$= \left\{ \sin\theta = t \Rightarrow dt = \cos\theta \, d\theta \right\} = \frac{1}{\sqrt{2}} \int (1 - t^2)\, dt = \frac{1}{\sqrt{2}} \left(t - \frac{t^3}{3} \right) + C$$

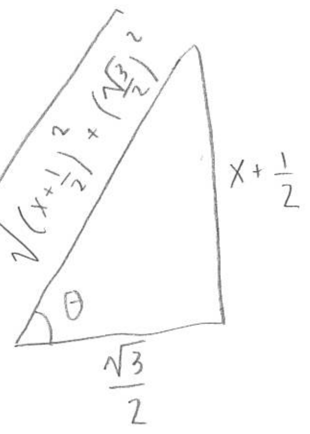
$$= \left\{ t = \sin\theta = \frac{\sqrt{2}\, x}{\sqrt{1+2x^2}} \right\} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}\, x}{\sqrt{1+2x^2}} - \frac{2\sqrt{2}\, x^3}{3(1+2x^2)^{3/2}} \right) + C$$

$$= \frac{3x(1+2x^2) - 2x^3}{3(1+2x^2)^{3/2}} + C = \frac{4x^3 + 3x}{3(1+2x^2)^{3/2}} + C$$

A.6.3.18. Beräkna $\int \frac{1}{x^2+x+1}\, dx$

Vi ser att $\frac{1}{x^2+x+1} = \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$

$$= \frac{1}{\left(\sqrt{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right)^2}$$



$$\sin\theta = \frac{x + \frac{1}{2}}{\sqrt{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}, \quad \cos\theta = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}$$

$$\tan\theta = \frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \Leftrightarrow x = \frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2} = \frac{1}{2}(\sqrt{3}\tan\theta - 1)$$

Vi deriverar $\tan\theta$-uttrycket.

$$\frac{dx}{d\theta} = \frac{\sqrt{3}}{2} \cdot \frac{1}{\cos^2\theta} \Rightarrow dx = \frac{\sqrt{3}}{2} \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx = \frac{4}{3}\int \cos^2\theta \cdot \underbrace{\frac{\sqrt{3}}{2}\cdot\frac{1}{\cos^2\theta}\,d\theta}_{dx}$$

$$= \frac{2}{\sqrt{3}}\int d\theta = \frac{2\theta}{\sqrt{3}} + C = \left\{ \tan\theta = \frac{1}{\sqrt{3}}(2x+1) \Rightarrow \theta = \arctan\left(\frac{2x+1}{\sqrt{3}}\right) \right\}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

6.5.2. Beräkna $\displaystyle\int_3^\infty \frac{1}{(2x-1)^{3/2}}\,dx$

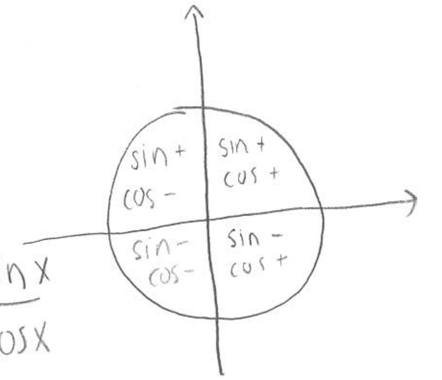
En generaliserad integral eftersom övre integrationsgränsen är obegränsad!

eftersom $\frac{d}{dx}(2x-1)^{-1/2} = -\frac{2}{2(2x-1)}$

$$\int_3^\infty \frac{1}{(2x-1)^{3/2}}\,dx = \lim_{R\to\infty}\int_3^R \frac{1}{(2x-1)^{3/2}}\,dx = \lim_{R\to\infty}\left[-\frac{1}{2\sqrt{2x-1}}\right]_3^R$$

dela med inre derivatan

$$= \lim_{R\to\infty} \underbrace{\frac{-1}{\sqrt{2R-1}}}_{\to 0} + \frac{1}{\sqrt{2\cdot 3-1}} = \frac{1}{\sqrt{5}}$$



$\tan x = \frac{\sin x}{\cos x}$

6.5.4. Beräkna $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2+1}\,dx$

$$\int_{-\infty}^{-1} \frac{1}{x^2+1}\,dx = \lim_{R\to-\infty}\int_R^{-1}\frac{1}{x^2+1}\,dx = \lim_{R\to-\infty}\Big[\arctan x\Big]_R^{-1}$$

$$= \arctan(-1) - \lim_{R\to-\infty}\arctan(R) = \left\{ \arctan x \text{ definierad för } x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \right\}$$

$$= \left\{ \sin -\frac{\pi}{4} = -\cos -\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \Rightarrow \arctan(-1) = -\frac{\pi}{4} \right\}$$

$$= -\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{4} \quad \text{eftersom} \quad \begin{matrix} \sin x \to -1 \\ \cos x \to 0 \end{matrix} \quad , \quad x \to -\frac{\pi}{2} .$$

6.5.18 Beräkna $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

$$\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \left\{ t = \ln x ,\ dt = \frac{1}{x}dx ,\ x = e \Rightarrow t = 1 ,\ x \to \infty \Rightarrow t \to \infty \right\}$$

$$= \int_1^\infty \frac{1}{t^2}\,dt = \lim_{R \to \infty} \int_1^R \frac{1}{t^2}\,dt = \lim_{R \to \infty} \left[-\frac{1}{t} \right]_1^R = 0 - (-1) = 1 .$$