Inledande matematik för E1, (TMV156), 090826

1. (a) In matrix form the system is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & -2 & 2 \end{array}\right).$$

The sequence of row operations

$$R_2 \mapsto R_2 - R_1, \quad R_2 \mapsto \frac{1}{2}R_2,$$

transforms the system to the echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right).$$

Thus we can choose z as a free variable. Back substitution gives, first of all,

$$y - 2z = 1 \Rightarrow y = 1 + 2z,$$

and then

$$x - y + 2z = 0 \Rightarrow x - (1+2z) + 2z = 0 \Rightarrow x = 1.$$

ANSWER : $\{(1, 1+2z, z) : z \in \mathbb{R}\}$.

(b) In polar form,

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

and hence, by De Moivre's theorem,

$$(-1+i)^{20} = 2^{10}\left(\cos 15\pi + i \sin 15\pi\right) = 2^{10}(-1) = -2^{10}.$$

(c) The numerator, being an absolute value, is always non-negative, and zero at $x = -2$. Hence, the quotient is positive if and only if the denominator is so, and this is the case when $x > 2$. Hence the quotient is non-negative for $x = -2$ and for $x > 2$.
ANSWER : $x \in \{-2\} \cup (2, \infty)$.

d When $\sin x$ attains its maximum (resp. minimum) value, then the numerator attains its maximum (resp. minimum) value and, simoultaneously, the denominator attains its minimum (resp. maximum) value. Hence the quotient also attains its maximum (resp. minimum) value here. Since the maximum (resp. minimum) value of $\sin x$ is $+1$ (resp. -1), the maximum (resp. minimum) value of f is $\frac{2+1}{3-2} = 3$ (resp. $\frac{2-1}{3+2} = \frac{1}{5}$.
ANSWER : Max. value is 3 and min. value is 1/5.

(e) **(i)** Both the numerator and denominator go to zero as $x \to 0$. Rewrite the quotient as

$$\frac{\ln(1+2x)}{2x} \cdot \frac{2x}{\sin x}$$

and observe that, as $x \to 0$, these two quotients go respectively to 1 and 2. Hence the limit is 2.
(ii) We can write

$$\left(1 + \frac{2}{x}\right)^{3x} = \left[\left(1 + \frac{2}{x}\right)^{x/2}\right]^6.$$

The inner function tends to e as $x \to \infty$, hence the whole thing tends to e^6.
(iii) Exponentials defeat polynomials as $x \to \infty$ and the denominator has the larger exponential function. Hence the quotient goes to zero.

(f) When $x = 2$ we have $y^3 + y = 2$ and one sees directly that $y = 1$. Implicit differentiation gives

$$(3y^2 + 1)y' = 1 \Rightarrow y' = \frac{1}{3y^2 + 1}.$$

At $x = 2$ we've already seen that $y = 1$ and hence $y'(2) = \frac{1}{3 \cdot 1^2 + 1} = \frac{1}{4}$.
A further implicit differentiation yields

$$y'' = \frac{-6yy'}{(3y^2 + 1)^2}.$$

At $x = 2$, we've seen that $y = 1$ and $y' = 1/4$, hence

$$y''(2) = \frac{-6 \cdot 1 \cdot \frac{1}{4}}{(3 \cdot 1^2 + 1)^2} = -\frac{3}{32}.$$

ANSWER : $y(2) = 1$, $y'(2) = 1/4$, $y''(2) = -3/32$.

2. (a) The equation of any parallel plane has the form $x + 2y + 3z = d$. Inserting $(1, 2, 3)$ we find that $d = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14$. Hence the plane in question is $x + 2y + 3z = 14$.

(b) Finding the intersection between the two planes means solving the given system of two equations. Let z be the free variable and one finds that the general solution is $x = 3 + z$, $y = 1 - x - z = 1 - (3 + z) - z = -2 - 2z$. Hence the intersection of the two planes is the set of points $(3 + z, -2 - 2z, z)$, where $z \in \mathbb{R}$. We can write this set in the parametric vector form

$$\{(3, -2, 0) + z \cdot (1, -2, 1) : z \in \mathbb{R}\}.$$

This is the equation of a line in the direction of $\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

(c) The equation of a plane with normal vector $(-3, 3, -4)$ has the form $-3x + 3y - 4z = d$. To find d, we just need one point in the plane, in other words, just one point of intersection between the planes $x + y - 2z = 6$ and $2x - y + z = 2$. Putting $x = 0$, say, we easily find that $(0, -10, -8)$ is a point of intersection. Hence $d = -3 \cdot 0 + 3 \cdot (-10) - 4 \cdot (-8) = 2$.

ANSWER : The plane is $-3x + 3y - 4z = 2$.

3. Funktionen $f(x) = \ln|x - 3| + \arctan x$ har definitionsmängd $\mathcal{D}(f) = [-1, 3)$ och är deriverbar på hela intervallet med $f'(x) = \frac{1}{x-3} + \frac{1}{x^2+1} = \frac{x^2+x-2}{(x-3)(x^2+1)} = \frac{(x+2)(x-1)}{(x-3)(x^2+1)}$. Vi ser att f' har ett nollställe $x = 1 \in \mathcal{D}(f)$, att $f'(x) > 0$ i $[-1, 1)$, $f'(x) < 0$ i $(1, 3)$ och $f(x) \to -\infty$ då $x \to 3$. Detta ger ett globalt maximum $f(1) = \ln 2 + \frac{\pi}{4}$. Då f är kontinuerlig får vi att f antar alla reella värden $\leq f(1)$. Alltså får vi att värdemängden är $\mathcal{R}(f) = (-\infty, \ln 2 + \frac{\pi}{4}]$.

4. *Step 0* : Search for any obvious symmetries.

There aren't any.

Step 1 : Investigate behaviour as $x \to \pm\infty$.

Noting that $f(x)$ is a polynomial of odd degree, we have that

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty.$$

Step 2 : Investigate the domain and possible vertical asymptotes.

There aren't any vertical asymptotes (polynomials are continuous everywhere).

Step 3 : Find and classify the critical points.

Differentiate to get

$$f'(x) = 60x^4 - 60x^3 - 60x^2 + 60x.$$

Hence we must solve the equation

$$60x^4 - 60x^3 - 60x^2 + 60x = 0 \Rightarrow 60x(x^3 - x^2 - x + 1) = 0.$$

Hence $x = 0$ is one critical point. Furthermore, observe that

$$x^3 - x^2 - x + 1 = (x^3 + 1) - x(x + 1)$$
$$= (x + 1)(x^2 - x + 1) - x(x + 1) = (x + 1)(x^2 - 2x + 1) = (x + 1)(x - 1)^2.$$

Hence the other critical points are at $x = \pm 1$. Then, since

$$f(x) = 60x(x + 1)(x - 1)^2,$$

a sign table easily shows that $x = -1$ is a local maximum, $x = 0$ is a local minimum and $x = 1$ is an inflection point.

We can now draw the graph.(A picture should be included here).

5. a) Antag att f är definierad för $I \setminus \{a\}$ där I är ett intervall runt a. Då gäller att

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0 \text{ sådant att } 0 < |x - a| < \delta \Rightarrow |f(x) - A| < \varepsilon.$$

b) Låt $\varepsilon > 0$ vara givet och tag $\delta = \varepsilon/3$. Då gäller att om $0 < |x - 1| < \delta$ så har vi $|3x + 1 - 4| = 3|x - 1| < 3\delta = 3(\varepsilon/3) = \varepsilon$; d v s definitionen av att $\lim_{x \to 1} 3x + 1 = 4$ är uppfylld.

c) Låt $\varepsilon > 0$ vara givet och tag $\delta = \varepsilon$. Då gäller att om $0 < |x - 0| < \delta$ så har vi $|x \sin \frac{1}{x} - 0| = |x||\sin \frac{1}{x}| \leq |x| = |x - 0| < \delta = \varepsilon$. D v s $\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0$ (nämligen $\delta = \varepsilon$) sådant att $0 < |x - 0| < \delta \Rightarrow |x \sin \frac{1}{x} - 0| < \varepsilon$; vilket alltså är vad vi menar med att $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

6. (a) True. This is the contrapositive of Theorem 1 in Section 2.3 (of the seventh edition).
   (b) True. It's one-to-one (since $f'(x) = 3x^2 + 1 > 0$ for all x) and onto (since f is a polynomial of odd degree).
   (c) False. The given combination will always be perpendicular to $\boldsymbol{u}$, but could lie at any angle whatsoever to $\boldsymbol{v}$.
   (d) False, since the range of arctan is only the interval $(-\pi/2, \pi/2)$.
   (e) True.
   (f) False. For example, suppose $f(x) = -1$ for $x < 0$ and $f(x) = +1$ for $x \geq 0$. Then f is not even continuous at $x = 0$, hence not differentiable either (see part (a)). But $|f(x)|$ is a constant function, hence differentiable everywhere.

7. (a)

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

   (b) Theorem 3 in Section 2.3 (seventh edition).