KU LVS 2017

(1.) Bestäm determinanten av

$$A = \begin{bmatrix} 1 & 3 & 2 & -7 \\ 2 & 9 & -2 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Lösning:

Egenskaper hos determinanter för $n \times n$-mat.

1. $\det AB = \det A \cdot \det B$

2a) Addition av en multipel av en rad till en annan ändrar ej det.

b) Byte av rad $= (-1) \cdot \det$

c) Mult av en rad i A med k för att få $B \Rightarrow \det B = k \cdot \det A$

3. Om A är diag $\Rightarrow \det A = \prod_{i=1}^{n} a_{ii}$

$$\det A = \begin{vmatrix} 1 & 3 & 2 & -7 \\ 2 & 9 & -2 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 1 & 5 \end{vmatrix} (-2) \overset{2a}{=} \begin{vmatrix} 1 & 3 & 2 & -7 \\ 0 & 3 & -6 & 15 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 1 & 5 \end{vmatrix} (-1)$$

$$\overset{2a}{=} \begin{vmatrix} 1 & 3 & 2 & -7 \\ 0 & 3 & -6 & 15 \\ 0 & 0 & 3 & -12 \\ 0 & 0 & 1 & 5 \end{vmatrix} \cdot \frac{1}{3} = \underbrace{\begin{vmatrix} 1 & 3 & 2 & -7 \\ 0 & 3 & -6 & 15 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 5 \end{vmatrix}}_{= \frac{1}{3} \cdot \det A} \cdot 3 \ (-1) \overset{2a}{=} 3 \begin{vmatrix} 1 & 3 & 2 & -7 \\ 0 & 3 & -6 & 15 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 9 \end{vmatrix}$$

$$\overset{3}{=} 3 \cdot (1 \cdot 3 \cdot 1 \cdot 9) = 9 \cdot 9 = 81$$

(Matlab ger också 81.)

Kofaktorexpansion

$$\det A = \begin{vmatrix} 1 & 3 & 2 & -7 \\ 2 & 9 & -2 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 1 & 5 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 9 & -2 & 1 \\ 3 & -3 & 3 \\ 0 & 1 & 5 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 & -7 \\ 3 & -3 & 3 \\ 0 & 1 & 5 \end{vmatrix}$$

$$= 1 \cdot \left(9 \begin{vmatrix} -3 & 3 \\ 1 & 5 \end{vmatrix} - 3 \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} \right)$$

$$- 2 \cdot \left(3 \begin{vmatrix} -3 & 3 \\ 1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & -7 \\ 1 & 5 \end{vmatrix} \right)$$

$$= 1 \cdot \left(9(-15-3) - 3(-10-1) \right)$$

$$- 2 \cdot \left(3(-15-3) - 3(10+7) \right)$$

$$= 1 \cdot (9 \cdot -18 + 33) - 2(3(-18) - 3 \cdot 17)$$

$$= -9 \cdot 18 + 33 + 6 \cdot 18 + 6 \cdot 17$$

$$= -3 \cdot 18 + 33 + 6 \cdot 10 + 6 \cdot 7$$

$$= 60 + 42 + 33 - 54 = 135 - 54$$

$$= \begin{array}{r} \overset{10}{\cancel{1}\,3\,5} \\ -\;5\,4 \\ \hline 8\,1 \end{array} = 81$$

(2.) Matrisen $A = \begin{bmatrix} 1 & 8 & 4 \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix}$ har

åtminstone egenvektorerna

$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ och $v_2 = \begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}$.

Bestäm alla egenvektorer och tillhörande egenvärden.

<u>Lösning</u>: Kar. ekv.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 8 & 4 \\ 0 & 7-\lambda & 0 \\ 2 & -8 & -1-\lambda \end{vmatrix}$$

(Använd kofaktorexp. med andra raden)

$$= (7-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{vmatrix}$$

$(a-b)(a+b)$

$a^2 - b^2$

$\cancel{\lambda^2 - 9}$

$$= (7-\lambda)\left((1-\lambda)(-1-\lambda) - 8\right)$$

$$= (7-\lambda)\left(-1 - \cancel{\lambda} + \cancel{\lambda} + \lambda^2 - 8\right)$$

$$= (7-\lambda)(\lambda^2 - 9)$$

$$= (7-\lambda)(\lambda - 3)(\lambda + 3) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = 3, \quad \lambda_3 = 7$$

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$$

Lös detta ~~sys~~ LES för ~~varje~~ λ_1 för att få egenvektorerna.

$\lambda_1 = -3$

$(A - \lambda_1 I)x = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & 8 & 4 & 0 \\ 0 & 10 & 0 & 0 \\ 2 & -8 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 24 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 2 & -8 & 2 & 0 \end{array}\right] \Rightarrow x_2 = 0$$

$x_1 = -x_3$, x_3 = fri

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

För $\lambda = -3$ är egenvektorn $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

För ~~resterande~~ att få fram egenvärden med given egenvektorer (använd v_1 & v_2)

$$Av_1 = \begin{bmatrix} 1 & 8 & 4 \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} = 3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{=v_1} \quad \lambda = 3$$

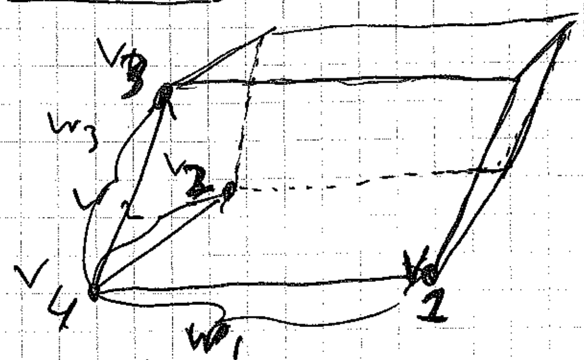
$$Av_2 = \begin{bmatrix} 1 & 8 & 4 \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 35 \\ -28 \end{bmatrix} = 7 \underbrace{\begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}}_{=v_2} \Rightarrow \lambda = 7$$

$\lambda = -3$	$\lambda = 3$	$\lambda = 7$
$v = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$	$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$	$v = \begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}$

(3.) Bestäm volymen av den parallellpiped som har de fyra närliggande hörnen

$$\underset{v_1}{\begin{bmatrix}2\\3\\0\end{bmatrix}},\quad \underset{v_2}{\begin{bmatrix}-1\\2\\1\end{bmatrix}},\quad \underset{v_3}{\begin{bmatrix}4\\3\\7\end{bmatrix}} \quad \text{och} \quad \underset{v_4}{\begin{bmatrix}2\\1\\0\end{bmatrix}}$$

Lösning



$$V = \left| w_3 \cdot (w_1 \times w_2) \right| \quad (\text{t.ex.})$$

Låt $w_1 = v_1 - \cancel{v_4}\, v_4$, $w_2 = v_2 - v_4$, $w_3 = v_3 - v_4$

$$w_1 = \begin{bmatrix}2-2\\3-1\\0-0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

$$w_2 = \begin{bmatrix}-1-2\\2-1\\1-0\end{bmatrix} = \begin{bmatrix}-3\\1\\1\end{bmatrix}$$

$$w_3 = \begin{bmatrix}4-2\\3-1\\7-0\end{bmatrix} = \begin{bmatrix}2\\2\\7\end{bmatrix}$$

$$w_1 \times w_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ -3 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(2) + \hat{j}(-0) + \hat{k}(6)$$

$$= \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

$$V = |w_3 \circ (w_1 \times w_2)| = \text{abs}\left(\begin{bmatrix} 2 & 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \right)$$

$$= |4 + 42| = 46$$

4. Bestäm egenvärden och egenvektorer till rotationsmatrisen

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lösning:

$$0 = \det(M - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (1-\lambda)\left((\cos\theta - \lambda)^2 + \sin^2\theta \right)$$

$$= (1-\lambda)\left(\cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta \right)$$

$$= \underbrace{(1-\lambda)}_{\substack{=0 \\ \text{om } \lambda = 1}} \underbrace{\left(\lambda^2 - 2\lambda\cos\theta + 1 \right)}_{=0}$$

$$\Rightarrow \lambda = \cos\theta \pm \sqrt{\underbrace{\cos^2\theta - 1}_{= -\sin^2\theta}}$$

$$= \cos\theta \pm i\sin\theta$$

$$\lambda_1 = 1, \quad \lambda_2 = \cos\theta + i\sin\theta, \quad \lambda_3 = \cos\theta - i\sin\theta$$

7.

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$$

Løs dette LES for å få x.

$\underline{\lambda_1 = 1}$

$$\Rightarrow \left[\begin{array}{ccc|c} \cos\theta - 1, & -\sin\theta, & 0 & 0 \\ \sin\theta, & \cos\theta - 1, & 0 & 0 \\ 0, & 0, & 0 & 0 \end{array}\right] \quad (-1) \quad x_3 \text{ fri}$$

$$\begin{pmatrix} \cos\theta\, x_1 - x_1 - \sin\theta\, x_2 = 0 \\ \sin\theta\, x_1 + \cos\theta\, x_2 - x_2 = 0 \end{pmatrix}$$

$$\left(\left[\begin{array}{ccc|c} \cos\theta - \sin\theta - 1, & -\sin\theta - (\cos\theta - 1), & 0 & 0 \\ \sin\theta, & \cos\theta - 1, & 0 & 0 \\ 0, & 0, & 0 & 0 \end{array}\right]\right)$$

$$(\cos\theta - \sin\theta - 1)x_1 = (\cos\theta + \sin\theta - 1)x_2$$

$$(\cos\theta - 1)x_1 = \sin\theta\, x_2$$

$$(\sin\theta)\, x_1 = -(\cos\theta - 1)x_2$$

$$x_1 = \frac{\sin\theta}{\cos\theta - 1} x_2 = -\frac{(\cos\theta - 1)}{\sin\theta} x_2$$

$$\Rightarrow \sin^2\theta = -(\cos\theta - 1)^2 = -(\cos^2\theta - 2\cos\theta + 1)$$

$$\Rightarrow \underbrace{\sin^2\theta + \cos^2\theta}_{=1} = 2\cos\theta + \cancel{1} \Rightarrow \theta = \pm\frac{\pi}{2} \pm \pi n,\ n \in \mathbb{N}$$

Men dette ska holde for godtycklig $\theta \Longrightarrow x_2 = 0 \Rightarrow x_1 = 0$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}, \quad \forall t \in \mathbb{R}.$$

$\underline{\lambda_2 = \cos\theta + i\sin\theta}$

$$\left[\begin{array}{ccc|c} -i\sin\theta & -\sin\theta & 0 & 0 \\ \sin\theta & -i\sin\theta & 0 & 0 \\ 0 & 0 & 1-\cos\theta-i\sin\theta & 0 \end{array}\right] \Rightarrow x_3 = 0$$

$$\cancel{\sim \left[\begin{array}{ccc|c} -i\sin\theta & -\sin\theta & 0 & 0 \\ \sin\theta - i\sin\theta & & 0 & 0 \\ 0 & 0 & 1-\cos\theta-i\sin\theta & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -\sin\theta & 0 & 0 & 0 \\ 0 & -2\sin\theta & 0 & 0 \\ 0 & 0 & 1-\cos\theta-i\sin\theta & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = 0 \\ \Rightarrow x_2 = 0 \\ \Rightarrow x_3 = 0 \end{array}}$$

$$v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{ej' OK egenvektor!}$$

$$\underbrace{-i\sin\theta}_{\text{im}}\ \underbrace{x_1}_{\text{Re}} = \underbrace{\sin\theta\, x_2}_{\text{reelt}} \Rightarrow x_1 = x_2 = 0$$

$$\lambda_3 = \cos\theta - i\sin\theta$$

$$\left[\begin{array}{ccc|c} i\sin\theta & -\sin\theta & 0 & 0 \\ \sin\theta & i\sin\theta & 0 & 0 \\ 0 & 0 & 1-\cos\theta - i\sin\theta & 0 \end{array}\right]$$

$\underbrace{i\sin\theta x_1}_{i\cdot m} = \underbrace{\sin\theta x_2}_{\text{Reell}}$

$\Rightarrow x_1 = x_2 = 0$

$$\cancel{\left[\begin{array}{ccc|c} \sin\theta & -\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 & 0 \\ 0 & 0 & 1+\ldots & 0 \end{array}\right]}$$

$\Rightarrow x_1 = 0$

$\Rightarrow x_2 = 0$

$\Rightarrow x_3 = 0$

$v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ej giltig egenvektor!

$\therefore \quad \lambda = 1, \quad v = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}, \forall t \in \mathbb{R}$

Detta går att tänka ut.

M roterar vektorer kring z-axeln

så $v = (0, 0, t)$ roteras ej betyder

$Mv = v \Rightarrow \lambda = 1$ och v egenvektor.

den enda som ej roteras.