Physical Modeling in MATLAB®

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The original form of this book is LaTeX source code. Compiling this code has the effect of generating a device-independent representation of a textbook, which can be converted to other formats and printed.

This book was typeset by the author using latex, dvips and ps2pdf, among other free, open-source programs. The LaTeX source for this book is available from `http://greenteapress.com/matlab`.

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Preface

Most books that use MATLAB are aimed at readers who know how to program. This book is for people who have never programmed before.

As a result, the order of presentation is unusual. The book starts with scalar values and works up to vectors and matrices very gradually. This approach is good for beginning programmers, because it is hard to understand composite objects until you understand basic programming semantics. But there are problems:

- The MATLAB documentation is written in terms of matrices, and so are the error messages. To mitigate this problem, the book explains the necessary vocabulary early and deciphers some of the messages that beginners find confusing.
- Many of the examples in the first half of the book are not idiomatic MATLAB. I address this problem in the second half by translating the examples into a more standard style.

The book puts a lot of emphasis on functions, in part because they are an important mechanism for controlling program complexity, and also because they are useful for working with MATLAB tools like `fzero` and `ode45`.

I assume that readers know calculus, differential equations, and physics, but not linear algebra. I explain the math as I go along, but the descriptions might not be enough for someone who hasn't seen the material before.

There are small exercises within each chapter, and a few larger exercises at the end of some chapters.

If you have suggestions and corrections, please send them to `downey@allendowney.com`.

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Contributor's list

The following are some of the people who have contributed to this book:

- Michael Lintz spotted the first (of many) typos.
- Kaelyn Stadtmueller reminded me of the importance of linking verbs.
- Roydan Ongie knows a matrix when he sees one (and caught a typo).
- Keerthik Omanakuttan knows that acceleration is not the second derivative of acceleration.
- Pietro Peterlongo pointed out that Binet's formula is an exact expression for the nth Fibonacci number, not an approximation.
- Li Tao pointed out several errors.
- Steven Zhang pointed out an error and a point of confusion in Chapter 11.
- Elena Oleynikova pointed out the "gotcha" that script file names can't have spaces.
- Kelsey Breseman pointed out that numbers as footnote markers can be confused with exponents, so now I am using symbols.
- Philip Loh sent me some updates for recent revisions of MATLAB.
- Harold Jaffe spotted a typo.
- Vidie Pong pointed out the problem with spaces in filenames.

Contents

Chapter 1

Variables and values

1.1 A glorified calculator

At heart, MATLAB is a glorified calculator. When you start MATLAB you will see a window entitled MATLAB that contains smaller windows entitled Current Directory, Command History and Command Window. The Command Window runs the MATLAB **interpreter**, which allows you to type MATLAB **commands**, then executes them and prints the result.

Initially, the Command Window contains a welcome message with information about the version of MATLAB you are running, followed by a chevron:

```
>>
```

which is the MATLAB **prompt**; that is, this symbol prompts you to enter a command.

The simplest kind of command is a mathematical **expression**, which is made up of **operands** (like numbers, for example) and **operators** (like the plus sign, `+`).

If you type an expression and then press Enter (or Return), MATLAB **evaluates** the expression and prints the result.

```
>> 2 + 1
ans = 3
```

Just to be clear: in the example above, MATLAB printed `>>`; I typed `2 + 1` and then hit Enter, and MATLAB printed `ans = 3`. And when I say "printed," I really mean "displayed on the screen," which might be confusing, but it's the way people talk.

An expression can contain any number of operators and operands. You don't have to put spaces between them; some people do and some people don't.

```
>> 1+2+3+4+5+6+7+8+9
ans = 45
```

Speaking of spaces, you might have noticed that MATLAB puts some space between ans = and the result. In my examples I will leave it out to save paper.

The other arithmetic operators are pretty much what you would expect. Subtraction is denoted by a minus sign, -; multiplication by an asterisk, * (sometimes pronounced “splat”); division by a forward slash /.

```
>> 2*3 - 4/5
ans = 5.2000
```

The order of operations is what you would expect from basic algebra: multiplication and division happen before addition and subtraction. If you want to override the order of operations, you can use parentheses.

```
>> 2 * (3-4) / 5
ans = -0.4000
```

When I added the parentheses I also changed the spacing to make the grouping of operands clearer to a human reader. This is the first of many style guidelines I will recommend for making your programs easier to read. Style doesn't change what the program does; the MATLAB interpreter doesn't check for style. But human readers do, and the most important human who will read your code is you.

And that brings us to the First Theorem of debugging:

> Readable code is debuggable code.

It is worth spending time to make your code pretty; it will save you time debugging!

The other common operator is exponentiation, which uses the ^ symbol, sometimes pronounced “carat” or “hat”. So 2 raised to the 16th power is

```
>> 2^16
ans = 65536
```

As in basic algebra, exponentiation happens before multiplication and division, but again, you can use parentheses to override the order of operations.

1.2 Math functions

MATLAB knows how to compute pretty much every math function you've heard of. It knows all the trigonometric functions; here's how you use them:

```
>> sin(1)
ans = 0.8415
```

This command is an example of a **function call**. The name of the function is `sin`, which is the usual abbreviation for the trigonometric sine. The value in parentheses is called the **argument**. All the trig functions in MATLAB work in radians.

Some functions take more than one argument, in which case they are separated by commas. For example, `atan2` computes the inverse tangent, which is the angle in radians between the positive x-axis and the point with the given y and x coordinates.

```
>> atan2(1,1)
ans = 0.7854
```

If that bit of trigonometry isn't familiar to you, don't worry about it. It's just an example of a function with multiple arguments.

MATLAB also provides exponential functions, like `exp`, which computes e raised to the given power. So `exp(1)` is just e.

```
>> exp(1)
ans = 2.7183
```

The inverse of `exp` is `log`, which computes the logarithm base e:

```
>> log(exp(3))
ans = 3
```

This example also demonstrates that function calls can be **nested**; that is, you can use the result from one function as an argument for another.

More generally, you can use a function call as an operand in an expression.

```
>> sqrt(sin(0.5)^2 + cos(0.5)^2)
ans = 1
```

As you probably guessed, `sqrt` computes the square root.

There are lots of other math functions, but this is not meant to be a reference manual. To learn about other functions, you should read the documentation.

1.3 Documentation

MATLAB comes with two forms of online documentation, `help` and `doc`.

The help command works from the Command Window; just type `help` followed by the name of a command.

```
>> help sin
 SIN    Sine of argument in radians.
    SIN(X) is the sine of the elements of X.

    See also asin, sind.
    Overloaded functions or methods (ones with the same name in other
       directories) help sym/sin.m

    Reference page in Help browser
       doc sin
```

Unfortunately, this documentation is not beginner-friendly.

One gotcha is that the name of the function appears in the help page in capital letters, but if you type it like that in MATLAB, you get an error:

```
>> SIN(1)
??? Undefined command/function 'SIN'.
```

Another problem is that the help page uses vocabulary you don't know yet. For example, "the elements of X" won't make sense until we get to vectors and matrices a few chapters from now.

The doc pages are usually better. If you type `doc sin`, a browser appears with more detailed information about the function, including examples of how to use it. The examples often use vectors and arrays, so they may not make sense yet, but you can get a preview of what's coming.

1.4 Variables

One of the features that makes MATLAB more powerful than a calculator is the ability to give a name to a value. A named value is called a **variable**.

MATLAB comes with a few predefined variables. For example*, the name `pi` refers to the mathematical quantity π, which is approximately

```
>> pi
ans = 3.1416
```

And if you do anything with complex numbers, you might find it convenient that both `i` and `j` are predefined as the square root of -1.

You can use a variable name anywhere you can use a number; for example, as an operand in an expression:

```
>> pi * 3^2
ans = 28.2743
```

or as an argument to a function:

```
>> sin(pi/2)
ans = 1

>> exp(i * pi)
ans = -1.0000 + 0.0000i
```

*Technically `pi` is a function, not a variable, but for now it's best to pretend.

As the second example shows, many MATLAB functions work with complex numbers. This example demonstrates Euler's Equality: $e^{i\pi} = -1$.

Whenever you evaluate an expression, MATLAB assigns the result to a variable named ans. You can use ans in a subsequent calculation as shorthand for "the value of the previous expression".

```
>> 3^2 + 4^2
ans = 25

>> sqrt(ans)
ans = 5
```

But keep in mind that the value of ans changes every time you evaluate an expression.

1.5 Assignment statements

You can create your own variables, and give them values, with an **assignment statement**. The assignment operator is the equals sign, =.

```
>> x = 6 * 7
x = 42
```

This example creates a new variable named x and assigns it the value of the expression 6 * 7. MATLAB responds with the variable name and the computed value.

In every assignment statement, the left side has to be a legal variable name. The right side can be any expression, including function calls.

Almost any sequence of lower and upper case letters is a legal variable name. Some punctuation is also legal, but the underscore, _, is the only commonly-used non-letter. Numbers are fine, but not at the beginning. Spaces are not allowed. Variable names are "case sensitive", so x and X are different variables.

```
>> fibonacci0 = 1;

>> LENGTH = 10;

>> first_name = 'allen'
first_name = allen
```

The first two examples demonstrate the use of the semi-colon, which suppresses the output from a command. In this case MATLAB creates the variables and assigns them values, but displays nothing.

The third example demonstrates that not everything in MATLAB is a number. A sequence of characters in single quotes is a **string**.

Although i, j and pi are predefined, you are free to reassign them. It is common to use i and j for other purposes, but it is probably not a good idea to change the value of pi!

1.6 Why variables?

The most common reasons to use variables are

- To avoid recomputing a value that is used repeatedly. For example, if you are performing computations involving e, you might want to compute it once and save the result.

  ```
  >> e = exp(1)
  e = 2.7183
  ```

- To make the connection between the code and the underlying mathematics more apparent. If you are computing the area of a circle, you might want to use a variable named `r`:

  ```
  >> r = 3
  r = 3

  >> area = pi * r^2
  area = 28.2743
  ```

  That way your code resembles the familiar formula πr^2.

- To break a long computation into a sequence of steps. Suppose you are evaluating a big, hairy expression like this:

  ```
  ans = ((x - theta) * sqrt(2 * pi) * sigma) ^ -1 * ...
  exp(-1/2 * (log(x - theta) - zeta)^2 / sigma^2)
  ```

  You can use an ellipsis to break the expression into multiple lines. Just type `...` at the end of the first line and continue on the next.

  But often it is better to break the computation into a sequence of steps and assign intermediate results to variables.

  ```
  shiftx = x - theta
  denom = shiftx * sqrt(2 * pi) * sigma
  temp = (log(shiftx) - zeta) / sigma
  exponent = -1/2 * temp^2
  ans = exp(exponent) / denom
  ```

  The names of the intermediate variables explain their role in the computation. `shiftx` is the value of `x` shifted by `theta`. It should be no surprise that `exponent` is the argument of `exp`, and `denom` ends up in the denominator. Choosing informative names makes the code easier to read and understand (see the First Theorem of Debugging).

1.7 Errors

It's early, but now would be a good time to start making errors. Whenever you learn a new feature, you should try to make as many errors as possible, as soon as possible.

When you make deliberate errors, you get to see what the error messages look like. Later, when you make accidental errors, you will know what the messages mean.

A common error for beginning programmers is leaving out the * for multiplication.

```
>> area = pi r^2
??? area = pi r^2
             |
Error: Unexpected MATLAB expression.
```

The error message indicates that, after seeing the operand `pi`, MATLAB was "expecting" to see an operator, like `*`. Instead, it got a variable name, which is the "unexpected expression" indicated by the vertical line, `|` (which is called a "pipe").

Another common error is to leave out the parentheses around the arguments of a function. For example, in math notation, it is common to write something like $\sin\pi$, but not in MATLAB.

```
>> sin pi
??? Function 'sin' is not defined for values of class 'char'.
```

The problem is that when you leave out the parentheses, MATLAB treats the argument as a string (rather than as an expression). In this case the `sin` function generates a reasonable error message, but in other cases the results can be baffling. For example, what do you think is going on here?

```
>> abs pi
ans =  112   105
```

There is a reason for this "feature", but rather than get into that now, let me suggest that you should *always* put parentheses around arguments.

This example also demonstrates the Second Theorem of Debugging:

> The only thing worse than getting an error message is *not* getting an error message.

Beginning programmers hate error messages and do everything they can to make them go away. Experienced programmers know that error messages are your friend. They can be hard to understand, and even misleading, but it is worth making some effort to understand them.

Here's another common rookie error. If you were translating the following mathematical expression into MATLAB:

$$\frac{1}{2\sqrt{\pi}}$$

You might be tempted to write something like this:

```
1 / 2 * sqrt(pi)
```

But that would be wrong. So very wrong.

1.8 Floating-point arithmetic

In mathematics, there are several kinds of numbers: integer, real, rational, irrational, imaginary, complex, etc. MATLAB only has one kind of number, called **floating-point**.

You might have noticed that MATLAB expresses values in decimal notation. So, for example, the rational number 1/3 is represented by the floating-point value

```
>> 1/3
ans = 0.3333
```

which is only approximately correct. It's not quite as bad as it seems; MATLAB uses more digits than it shows by default. You can change the `format` to see the other digits.

```
>> format long
>> 1/3
ans = 0.33333333333333
```

Internally, MATLAB uses the IEEE double-precision floating-point format, which provides about 15 significant digits of precision (in base 10). Leading and trailing zeros don't count as "significant" digits, so MATLAB can represent large and small numbers with the same precision.

Very large and very small values are displayed in scientific notation.

```
>> factorial(100)
ans = 9.332621544394410e+157
```

The `e` in this notation is *not* the transcendental number known as e; it is just an abbreviation for "exponent". So this means that 100! is approximately 9.33×10^{157}. The exact solution is a 158-digit integer, but we only know the first 16 digits.

You can enter numbers using the same notation.

```
>> speed_of_light = 3.0e8
speed_of_light = 300000000
```

Although MATLAB can handle large numbers, there is a limit. The predefined variables `realmax` and `realmin` contain the largest and smallest numbers that MATLAB can handle[†].

```
>> realmax
ans = 1.797693134862316e+308

>> realmin
ans = 2.225073858507201e-308
```

If the result of a computation is too big, MATLAB "rounds up" to infinity.

```
>> factorial(170)
ans = 7.257415615307994e+306

>> factorial(171)
ans = Inf
```

Division by zero also returns `Inf`, but in this case MATLAB gives you a warning because division by zero is usually considered undefined.

```
>> 1/0
Warning: Divide by zero.

ans = Inf
```

A warning is like an error message without teeth; the computation is allowed to continue. Allowing `Inf` to propagate through a computation doesn't always do what you expect, but if you are careful with how you use it, `Inf` can be quite useful.

For operations that are truly undefined, MATLAB returns `NaN`, which stands for "not a number".

```
>> 0/0
Warning: Divide by zero.

ans = NaN
```

1.9 Comments

Along with the commands that make up a program, it is useful to include comments that provide additional information about the program. The percent symbol `%` separates the comments from the code.

```
>> speed_of_light = 3.0e8     % meters per second
speed_of_light = 300000000
```

[†] The names of these variables are misleading; floating-point numbers are sometimes, wrongly, called "real".

The comment runs from the percent symbol to the end of the line. In this case it specifies the units of the value. In an ideal world, MATLAB would keep track of units and propagate them through the computation, but for now that burden falls on the programmer.

Comments have no effect on the execution of the program. They are there for human readers. Good comments make programs more readable, but bad comments are useless or (even worse) misleading.

Avoid comments that are redundant with the code:

```
>> x = 5        % assign the value 5 to x
```

Good comments provide additional information that is not in the code, like units in the example above, or the meaning of a variable:

```
>> p = 0         % position from the origin in meters
>> v = 100       % velocity in meters / second
>> a = -9.8      % acceleration of gravity in meters / second^2
```

If you use longer variable names, you might not need explanatory comments, but there is a tradeoff: longer code can become harder to read. Also, if you are translating from math that uses short variable names, it can be useful to make your program consistent with your math.

1.10 Glossary

interpreter: The program that reads and executes MATLAB code.

command: A line of MATLAB code executed by the interpreter.

prompt: The symbol the interpreter prints to indicate that it is waiting for you to type a command.

operator: One of the symbols, like * and +, that represent mathematical operations.

operand: A number or variable that appears in an expression along with operators.

expression: A sequence of operands and operators that specifies a mathematical computation and yields a value.

value: The numerical result of a computation.

evaluate: To compute the value of an expression.

order of operations: The rules that specify which operations in an expression are performed first.

function: A named computation; for example `log10` is the name of a function that computes logarithms in base 10.

call: To cause a function to execute and compute a result.

function call: A kind of command that executes a function.

argument: An expression that appears in a function call to specify the value the function operates on.

nested function call: An expression that uses the result from one function call as an argument for another.

variable: A named value.

assignment statement: A command that creates a new variable (if necessary) and gives it a value.

string: A value that consists of a sequence of characters (as opposed to a number).

floating-point: The kind of number MATLAB works with. All floating-point numbers can be represented with about 16 significant decimal digits (unlike mathematical integers and reals).

scientific notation: A format for typing and displaying large and small numbers; e.g. `3.0e8`, which represents 3.0×10^8 or 300,000,000.

comment: Part of a program that provides additional information about the program, but does not affect its execution.

1.11 Exercises

Exercise 1.1 *Write a MATLAB expression that evaluates the following math expression. You can assume that the variables* `mu`*,* `sigma` *and* `x` *already exist.*

$$\frac{e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2}}{\sigma\sqrt{2\pi}} \tag{1.1}$$

Note: you can't use Greek letters in MATLAB; when translating math expressions with Greek letters, it is common to write out the name of the letter (assuming you know it).

Chapter 2

Scripts

2.1 M-files

So far we have typed all of our programs “at the prompt,” which is fine if you are not writing more than a few lines. Beyond that, you will want to store your program in a **script** and then execute the script.

A script is a file that contains MATLAB code. These files are also called “M-files” because they use the extension `.m`, which is short for MATLAB.

You can create and edit scripts with any text editor or word processor, but the simplest way is by selecting New→Script from the File menu. A window appears running a text editor specially designed for MATLAB.

Type the following code in the editor

```
x = 5
```

and then press the (outdated) floppy disk icon, or select Save from the File menu. Either way, a dialog box appears where you can choose the file name and the directory where it should go. Change the name to `myscript.m` and leave the directory unchanged.

By default, MATLAB will store your script in a directory that is on the **search path**, which is the list of directories MATLAB searches for scripts.

Go back to the Command Window and type `myscript` (without the extension) at the prompt. MATLAB executes your script and displays the result.

```
>> myscript
x = 5
```

When you run a script, MATLAB executes the commands in the M-File, one after another, exactly as if you had typed them at the prompt.

If something goes wrong and MATLAB can’t find your script, you will get an error message like:

```
>> myscript
??? Undefined function or variable 'myscript'.
```

In this case you can either save your script again in a directory that is on the search path, or modify the search path to include the directory where you keep your scripts. You'll have to consult the documentation for the details (sorry!).

The filename can be anything you want, but you should try to choose something meaningful and memorable. You should be very careful to choose a name that is not already in use; if you do, you might accidentally replace one of MATLAB's functions with your own. Finally, the name of the file cannot contain spaces. If you create a file named `my script.m`, MATLAB doesn't complain until you try to run it:

```
>> my script
??? Undefined function or method 'my' for input arguments
of type 'char'.
```

The problem is that it is looking for a scipt named `my`. The problem is even worse if the first word of the filename is a function that exists. Just for fun, create a script named `abs val.m` and run it.

Keeping track of your scripts can be a pain. To keep things simple, for now, I suggest putting all of your scripts in the default directory.

Exercise 2.1 *The Fibonacci sequence, denoted F, is described by the equations $F_1 = 1$, $F_2 = 1$, and for $i \geq 3$, $F_i = F_{i-1} + F_{i-2}$. The elements of this sequence occur naturally in many plants, particularly those with petals or scales arranged in the form of a logarithmic spiral.*

The following expression computes the nth Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \tag{2.1}$$

Translate this expression into MATLAB and store your code in a file named `fibonacci1`. *At the prompt, set the value of* `n` *to 10 and then run your script. The last line of your script should assign the value of F_n to* `ans`. *(The correct value of F_{10} is 55).*

2.2 Why scripts?

The most common reasons to use scripts are:

- When you are writing more than a couple of lines of code, it might take a few tries to get everything right. Putting your code in a script makes it easier to edit than typing it at the prompt.

On the other hand, it can be a pain to switch back and forth between the Command Window and the Editor. Try to arrange your windows so you can see the Editor and the Command Window at the same time, and use the Tab key or the mouse to switch between them.

- If you choose good names for your scripts, you will be able to remember which script does what, and you might be able to reuse a script from one project to the next.
- If you run a script repeatedly, it is faster to type the name of the script than to retype the code!

Unfortunately, the great power of scripts comes with great responsibility, which is that you have to make sure that the code you are running is the code you think you are running.

First, whenever you edit your script, you have to save it before you run it. If you forget to save it, you will be running the old version.

Also, whenever you start a new script, start with something simple, like `x=5`, that produces a visible effect. Then run your script and confirm that you get what you expect. MATLAB comes with a lot of predefined functions. It is easy to write a script that has the same name as a MATLAB function, and if you are not careful, you might find yourself running the MATLAB function instead of your script.

Either way, if the code you are running is not the code you are looking at, you will find debugging a frustrating exercise! And that brings us to the Third Theorem of Debugging:

> You must always be 100% sure that the code you are running is the code you think you are running.

2.3 The workspace

The variables you create are stored in the **workspace**, which is a set of variables and their values. The `who` command prints the names of the variables in the workspace.

```
>> x=5;
>> y=7;
>> z=9;
>> who

Your variables are:

x  y  z
```

The `clear` command removes variables.

```
>> clear y
>> who

Your variables are:

x  z
```

To display the value of a variable, you can use the `disp` function.

```
>> disp(z)
     9
```

But it's easier to just type the variable name.

```
>> z
z = 9
```

(Strictly speaking, the name of a variable is an expression, so evaluating it should assign a value to `ans`, but MATLAB seems to handle this as a special case.)

2.4 More errors

Again, when you try something new, you should make a few mistakes on purpose so you'll recognize them later.

The most common error with scripts is to run a script without creating the necessary variables. For example, `fibonacci1` requires you to assign a value to `n`. If you don't:

```
>> fibonacci1
??? Undefined function or variable "n".

Error in ==> fibonacci1 at 4
diff = t1^(n+1) - t2^(n+1);
```

The details of this message might be different for you, depending on what's in your script. But the general idea is that `n` is undefined. Notice that MATLAB tells you what line of your program the error is in, and displays the line.

This information can be helpful, but beware! MATLAB is telling you where the error was discovered, not where the error is. In this case, the error is not in the script at all; it is, in a sense, in the workspace.

Which brings us to the Fourth Theorem of Debugging:

> Error messages tell you where the problem was discovered, not where it was caused.

The object of the game is to find the cause and fix it—not just to make the error message go away.

2.5 Pre- and post-conditions

Every script should contain a comment that explains what it does, and what the requirements are for the workspace. For example, I might put something like this at the beginning of fibonacci1:

```
% Computes the nth Fibonacci number.
% Precondition: you must assign a value to n before running
% this script.  Postcondition: the result is stored in ans.
```

A **precondition** is something that must be true, when the script starts, in order for it to work correctly. A **postcondition** is something that will be true when the script completes.

If there is a comment at the beginning of a script, MATLAB assumes it is the documentation for the script, so if you type help fibonacci1, you get the contents of the comment (without the percent signs).

```
>> help fibonacci1
  Computes the nth Fibonacci number.
  Precondition: you must assign a value to n before running
  this script.  Postcondition: the result is stored in ans.
```

That way, scripts that you write behave just like predefined scripts. You can even use the doc command to see your comment in the Help Window.

2.6 Assignment and equality

In mathematics the equals sign means that the two sides of the equation have the same value. In MATLAB an assignment statement *looks* like a mathematical equality, but it's not.

One difference is that the sides of an assignment statement are not interchangeable. The right side can be any legal expression, but the left side has to be a variable, which is called the **target** of the assignment. So this is legal:

```
>> y = 1;
>> x = y+1
x = 2
```

But this is not:

```
>> y+1 = x
??? y+1 = x
        |
Error: The expression to the left of the equals sign is not a valid
target for an assignment.
```

In this case the error message is pretty helpful, as long as you know what a "target" is.

Another difference is that an assignment statement is only temporary, in the following sense. When you assign `x = y+1`, you get the *current* value of `y`. If `y` changes later, `x` does not get updated.

A third difference is that a mathematical equality is a statement that may or may not be true. For example, $y = y + 1$ is a statement that happens to be false for all real values of y. In MATLAB, `y = y+1` is a sensible and useful assignment statement. It reads the current value of `y`, adds one, and replaces the old value with the new value.

```
>> y = 1;
>> y = y+1
y = 2
```

When you read MATLAB code, you might find it helpful to pronounce the equals sign "gets" rather than "equals." So `x = y+1` is pronounced "`x` gets the value of `y` plus one."

To test your understanding of assignment statements, try this exercise:

Exercise 2.2 *Write a few lines of code that swap the values of* `x` *and* `y`*. Put your code in a script called* `swap` *and test it.*

2.7 Incremental development

When you start writing scripts that are more than a few lines, you might find yourself spending more and more time debugging. The more code you write before you start debugging, the harder it is to find the problem.

Incremental development is a way of programming that tries to minimize the pain of debugging. The fundamental steps are

1. Always start with a working program. If you have an example from a book or a program you wrote that is similar to what you are working on, start with that. Otherwise, start with something you *know* is correct, like `x=5`. Run the program and confirm that you are running the program you think you are running.

   This step is important, because in most environments there are lots of little things that can trip you up when you start a new project. Get them out of the way so you can focus on programming.

2. Make one small, testable change at a time. A "testable" change is one that displays something on the screen (or has some other effect) that you can check. Ideally, you should know what the correct answer is, or be able to check it by performing another computation.

3. Run the program and see if the change worked. If so, go back to Step 2. If not, you will have to do some debugging, but if the change you made was small, it shouldn't take long to find the problem.

When this process works, you will find that your changes usually work the first time, or the problem is obvious. That's a good thing, and it brings us to the Fifth Theorem of Debugging:

> The best kind of debugging is the kind you don't have to do.

In practice, there are two problems with incremental development:

- Sometimes you have to write extra code to generate visible output that you can check. This extra code is called **scaffolding** because you use it to build the program and then remove it when you are done. But time you save on debugging is almost always worth the time you spend on scaffolding.
- When you are getting started, it is usually not obvious how to choose the steps that get from `x=5` to the program you are trying to write. There is an extended example in Section 5.7.

If you find yourself writing more than a few lines of code before you start testing, and you are spending a lot of time debugging, you should try incremental development.

2.8 Unit testing

In large software projects, **unit testing** is the process of testing software components in isolation before putting them together.

The programs we have seen so far are not big enough to need unit testing, but the same principle applies when you are working with a new function or a new language feature for the first time. You should test it in isolation before you put it into your program.

For example, suppose you know that `x` is the sine of some angle and you want to find the angle. You find the MATLAB function `asin`, and you are pretty sure it computes the inverse sine function. Pretty sure is not good enough; you want to be very sure.

Since we know $\sin 0 = 0$, we could try

```
>> asin(0)
ans = 0
```

which is correct. Also, we know that the sine of 90 degrees is 1, so if we try `asin(1)`, we expect the answer to be 90, right?

```
>> asin(1)
ans = 1.5708
```

Oops. We forgot that the trig functions in MATLAB work in radians, not degrees. So the correct answer is $\pi/2$, which we can confirm by dividing through by `pi`:

```
>> asin(1) / pi
ans = 0.5000
```

With this kind of unit testing, you are not really checking for errors in MATLAB, you are checking your understanding. If you make an error because you are confused about how MATLAB works, it might take a long time to find, because when you look at the code, it looks right.

Which brings us to the Sixth Theorem of Debugging:

> The worst bugs aren't in your code; they are in your head.

2.9 Glossary

M-file: A file that contains a MATLAB program.

script: An M-file that contains a sequence of MATLAB commands.

search path: The list of directories where MATLAB looks for M-files.

workspace: A set of variables and their values.

precondition: Something that must be true when the script starts, in order for it to work correctly.

postcondition: Something that will be true when the script completes.

target: The variable on the left side of an assignment statement.

incremental development: A way of programming by making a series of small, testable changes.

scaffolding: Code you write to help you program or debug, but which is not part of the finished program.

unit testing: A process of testing software by testing each component in isolation.

2.10 Exercises

Exercise 2.3 *Imagine that you are the owner of a car rental company with two locations, Albany and Boston. Some of your customers do "one-way rentals," picking up a car in Albany and returning it in Boston, or the other way around. Over time, you have observed that each week 5% of the cars in Albany are* *dropped off in Boston, and 3% of the cars in Boston get dropped off in Albany. At the beginning of the year, there are 150 cars at each location.*

Write a script called `car_update` *that updates the number of cars in each location from one week to the next. The precondition is that the variables* `a` *and* `b` *contain the number of cars in each location at the beginning of the week. The postcondition is that* `a` *and* `b` *have been modified to reflect the number of cars that moved.*

To test your program, initialize `a` *and* `b` *at the prompt and then execute the script. The script should display the updated values of* `a` *and* `b`*, but not any intermediate variables.*

Note: cars are countable things, so `a` *and* `b` *should always be integer values. You might want to use the* `round` *function to compute the number of cars that move during each week.*

If you execute your script repeatedly, you can simulate the passage of time from week to week. What do you think will happen to the number of cars? Will all the cars end up in one place? Will the number of cars reach an equilibrium, or will it oscillate from week to week?

In the next chapter we will see how to execute your script automatically, and how to plot the values of `a` *and* `b` *versus time.*

Chapter 3

Loops

3.1 Updating variables

In Exercise 2.3, you might have been tempted to write something like

```
a = a - 0.05*a + 0.03*b
b = b + 0.05*a - 0.03*b
```

But that would be wrong, so very wrong. Why? The problem is that the first line changes the value of a, so when the second line runs, it gets the old value of b and the new value of a. As a result, the change in a is not always the same as the change in b, which violates the principle of Conversation of Cars!

One solution is to use temporary variables anew and bnew:

```
anew = a - 0.05*a + 0.03*b
bnew = b + 0.05*a - 0.03*b
a = anew
b = bnew
```

This has the effect of updating the variables "simultaneously;" that is, it reads both old values before writing either new value.

The following is an alternative solution that has the added advantage of simplifying the computation:

```
atob = 0.05*a - 0.03*b
a = a - atob
b = b + atob
```

It is easy to look at this code and confirm that it obeys Conversation of Cars. Even if the value of atob is wrong, at least the total number of cars is right. And that brings us to the Seventh Theorem of Debugging:

> The best way to avoid a bug is to make it impossible.

In this case, removing redundancy also eliminates the opportunity for a bug.

3.2 Kinds of error

There are four kinds of error:

Syntax error: You have written a MATLAB command that cannot execute because it violates one of the rules of syntax. For example, you can't have two operands in a row without an operator, so `pi r^2` contains a syntax error. When MATLAB finds a syntax error, it prints an error message and stops running your program.

Runtime error: Your program starts running, but something goes wrong along the way. For example, if you try to access a variable that doesn't exist, that's a runtime error. When MATLAB detects the problem, it prints an error message and stops.

Logical error: Your program runs without generating any error messages, but it doesn't do the right thing. The problem in the previous section, where we changed the value of `a` before reading the old value, is a logical error.

Numerical error: Most computations in MATLAB are only approximately right. Most of the time the errors are small enough that we don't care, but in some cases the roundoff errors are a problem.

Syntax errors are usually the easiest. Sometimes the error messages are confusing, but MATLAB can usually tell you where the error is, at least roughly.

Run time errors are harder because, as I mentioned before, MATLAB can tell you where it detected the problem, but not what caused it.

Logical errors are hard because MATLAB can't help at all. Only you know what the program is supposed to do, so only you can check it. From MATLAB's point of view, there's nothing wrong with the program; the bug is in your head!

Numerical errors can be tricky because it's not clear whether the problem is your fault. For most simple computations, MATLAB produces the floating-point value that is closest to the exact solution, which means that the first 15 significant digits should be correct. But some computations are ill-conditioned, which means that even if your program is correct, the roundoff errors accumulate and the number of correct digits can be smaller. Sometimes MATLAB can warn you that this is happening, but not always! Precision (the number of digits in the answer) does not imply accuracy (the number of digits that are right).

3.3 Absolute and relative error

There are two ways of thinking about numerical errors, called **absolute** and **relative**.

An absolute error is just the difference between the correct value and the approximation. We usually write the magnitude of the error, ignoring its sign, because it doesn't matter whether the approximation is too high or too low.

For example, we might want to estimate 9! using the formula $\sqrt{18\pi}(9/e)^9$. The exact answer is $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880$. The approximation is $359,536.87$. The absolute error is 3,343.13.

At first glance, that sounds like a lot—we're off by three thousand—but it is worth taking into account the size of the thing we are estimating. For example, $3000 matters a lot if we are talking about my annual salary, but not at all if we are talking about the national debt.

A natural way to handle this problem is to use relative error, which is the error expressed as a fraction (or percentage) of the exact value. In this case, we would divide the error by 362,880, yielding .00921, which is just less than 1%. For many purposes, being off by 1% is good enough.

3.4 for loops

A **loop** is a part of a program that executes repeatedly; a **for loop** is the kind of loop that uses the `for` statement.

The simplest use of a `for` loop is to execute one or more lines a fixed number of times. For example, in the last chapter we wrote a script named `car_update` that simulates one week in the life of a rental car company. To simulate an entire year, we have to run it 52 times:

```
for i=1:52
    car_update
end
```

The first line looks like an assignment statement, and it *is* like an assignment statement, except that it runs more than once. The first time it runs, it creates the variable `i` and assigns it the value 1. The second time, `i` gets the value 2, and so on, up to 52.

The colon operator, `:`, specifies a **range** of integers. In the spirit of unit testing, you can create a range at the prompt:

```
>> 1:5
ans =  1     2     3     4     5
```

The variable you use in the for statement is called the **loop variable**. It is a common convention to use the names `i`, `j` and `k` as loop variables.

The statements inside the loop are called the **body**. By convention, they are indented to show that they are inside the loop, but the indentation does not actually affect the execution of the program. The end of the loop is officially marked by the `end` statement.

To see the loop in action you can run a loop that displays the loop variable:

```
>> for i=1:5
    i
end

i = 1
i = 2
i = 3
i = 4
i = 5
```

As this example shows, you *can* run a for loop from the command line, but it's much more common to put it in a script.

Exercise 3.1 *Create a script named* `car_loop` *that uses a* `for` *loop to run* `car_update` *52 times. Remember that before you run* `car_update`*, you have to assign values to* `a` *and* `b`*. For this exercise, start with the values* `a = 150` *and* `b = 150`*.*

If everything goes smoothly, your script will display a long stream of numbers on the screen. But it is probably too long to fit, and even if it fit, it would be hard to interpret. A graph would be much better!

3.5 plotting

`plot` is a versatile function for plotting points and lines on a two-dimensional graph. Unfortunately, it is so versatile that it can be hard to use (and hard to read the documentation!). We will start simple and work our way up.

To plot a single point, type

```
>> plot(1, 2)
```

A Figure Window should appear with a graph and a single, blue dot at x position 1 and y position 2. To make the dot more visible, you can specify a different shape:

```
>> plot(1, 2, 'o')
```

The letter in single quotes is a string that specifies how the point should be plotted. You can also specify the color:

```
>> plot(1, 2, 'ro')
```

`r` stands for red; the other colors include **g**reen, **b**lue, **c**yan, **m**agenta, **y**ellow and blac**k**. Other shapes include `+`, `*`, `x`, `s` (for square), `d` (for diamond), and `^` (for a triangle).

When you use `plot` this way, it can only plot one point at a time. If you run `plot` again, it clears the figure before making the new plot. The `hold` command lets you override that behavior. `hold on` tells MATLAB not to clear the figure when it makes a new plot; `hold off` returns to the default behavior.

Try this:

```
>> hold on
>> plot(1, 1, 'o')
>> plot(2, 2, 'o')
```

You should see a figure with two points. MATLAB scales the plot automatically so that the axes run from the lowest value in the plot to the highest. So in this example the points are plotted in the corners.

Exercise 3.2 *Modify* `car_loop` *so that each time through the loop it plots the value of* `a` *versus the value of* `i`*.*

Once you get that working, modify it so it plots the values of `a` *with red circles and the values of* `b` *with blue diamonds.*

One more thing: if you use `hold on` *to prevent MATLAB from clearing the figure, you might want to clear the figure yourself, from time to time, with the command* `clf`*.*

3.6 Sequences

In mathematics a **sequence** is a set of numbers that corresponds to the positive integers. The numbers in the sequence are called **elements**. In math notation, the elements are denoted with subscripts, so the first element of the series A is A_1, followed by A_2, and so on.

`for` loops are a natural way to compute the elements of a sequence. As an example, in a geometric sequence, each element is a constant multiple of the previous element. As a more specific example, let's look at the sequence with $A_1 = 1$ and the ratio $A_{i+1} = A_i/2$, for all i. In other words, each element is half as big as the one before it.

The following loop computes the first 10 elements of A:

```
a = 1
for i=2:10
    a = a/2
end
```

Each time through the loop, we find the next value of `a` by dividing the previous value by 2. Notice that the loop range starts at 2 because the initial value of `a` corresponds to A_1, so the first time through the loop we are computing A_2.

Each time through the loop, we replace the previous element with the next, so at the end, `a` contains the 10th element. The other elements are displayed on the screen, but they are not saved in a variable. Later, we will see how to save all of the elements of a sequence in a vector.

This loop computes the sequence **recurrently**, which means that each element depends on the previous one. For this sequence it is also possible to compute the ith element **directly**, as a function of i, without using the previous element. In math notation, $A_i = A_1 r^{i-1}$.

Exercise 3.3 *Write a script named* `sequence` *that uses a loop to compute elements of A directly.*

3.7 Series

In mathematics, a **series** is the sum of the elements of a sequence. It's a terrible name, because in common English, "sequence" and "series" mean pretty much the same thing, but in math, a sequence is a set of numbers, and a series is an expression (a sum) that has a single value. In math notation, a series is often written using the summation symbol $\sum$.

For example, the sum of the first 10 elements of A is

$$\sum_{i=1}^{10} A_i$$

A `for` loop is a natural way to compute the value of this series:

```
A1 = 1;
total = 0;
for i=1:10
    a = A1 * 0.5^(i-1);
    total = total + a;
end
ans = total
```

`A1` is the first element of the sequence, so each time through the loop `a` is the ith element.

The way we are using `total` is sometimes called an **accumulator**; that is, a variable that accumulates a result a little bit at a time. Before the loop we initialize it to 0. Each time through the loop we add in the ith element. At the end of the loop `total` contains the sum of the elements. Since that's the value we were looking for, we assign it to `ans`.

Exercise 3.4 *This example computes the terms of the series directly; as an exercise, write a script named* `series` *that computes the same sum by computing the elements recurrently. You will have to be careful about where you start and stop the loop.*

3.8 Generalization

As written, the previous example always adds up the first 10 elements of the sequence, but we might be curious to know what happens to `total` as we increase the number of terms in the series. If you have studied geometric series, you might know that this series converges on 2; that is, as the number of terms goes to infinity, the sum approaches 2 asymptotically.

To see if that's true for our program, we could replace the constant, 10, with a variable named `n`:

```
A1 = 1;
total = 0;
for i=1:n
    a = A1 * 0.5^(i-1);
    total = total + a;
end
ans = total
```

Now the script can compute any number of terms, with the precondition that you have to set `n` before you execute the script. Here's how you could run it with different values of `n`:

```
>> n=10; series

total = 1.99804687500000

>> n=20; series

total = 1.99999809265137

>> n=30; series

total = 1.99999999813735

>> n=40; series

total = 1.99999999999818
```

It sure looks like it's converging on 2.

Replacing a constant with a variable is called **generalization**. Instead of computing a fixed, specific number of terms, the new script is more general; it can compute any number of terms.

This is an important idea we will come back to when we talk about functions.

3.9 Glossary

absolute error: The difference between an approximation and an exact answer.

relative error: The difference between an approximation and an exact answer, expressed as a fraction or percentage of the exact answer.

loop: A part of a program that runs repeatedly.

loop variable: A variable, defined in a `for` statement, that gets assigned a different value each time through the loop.

range: The set of values assigned to the loop variable, often specified with the colon operator; for example `1:5`.

body: The statements inside the for loop that are run repeatedly.

sequence: In mathematics, a set of numbers that correspond to the positive integers.

element: A member of the set of numbers in a sequence.

recurrently: A way of computing the next element of a sequence based on previous elements.

directly: A way of computing an element in a sequence without using previous elements.

series: The sum of the elements in a sequence.

accumulator: A variable that is used to accumulate a result a little bit at a time.

generalization: A way to make a program more versatile, for example by replacing a specific value with a variable that can have any value.

3.10 Exercises

Exercise 3.5 *We have already seen the Fibonacci sequence, F, which is defined recurrently as*

$$F_i = F_{i-1} + F_{i-2}$$

In order to get started, you have to specify the first two elements, but once you have those, you can compute the rest. The most common Fibonacci sequence starts with $F_1 = 1$ and $F_2 = 1$.

Write a script called `fibonacci2` *that uses a for loop to compute the first 10 elements of this Fibonacci sequence. As a postcondition, your script should assign the 10th element to* `ans`*.*

Now generalize your script so that it computes the nth element for any value of `n`*, with the precondition that you have to set* `n` *before you run the script. To keep things simple for now, you can assume that* `n` *is greater than 2.*

Hint: you will have to use two variables to keep track of the previous two elements of the sequence. You might want to call them `prev1` *and* `prev2`*. Initially,* `prev1 =` F_1 *and* `prev2 =` F_2*. At the end of the loop, you will have to update* `prev1` *and* `prev2`*; think carefully about the order of the updates!*

Exercise 3.6 *Write a script named* `fib_plot` *that loops* `i` *through a range from 1 to 20, uses* `fibonacci2` *to compute Fibonacci numbers, and plots* F_i *for each* i *with a series of red circles.*

Chapter 4

Vectors

4.1 Checking preconditions

Some of the loops in the previous chapter don't work if the value of `n` isn't set correctly before the loop runs. For example, this loop computes the sum of the first `n` elements of a geometric sequence:

```
A1 = 1;
total = 0;
for i=1:n
    a = A1 * 0.5^(i-1);
    total = total + a;
end
ans = total
```

It works for any positive value of `n`, but what if `n` is negative? In that case, you get:

```
total = 0
```

Why? Because the expression `1:-1` means "all the numbers from 1 to -1, counting up by 1." It's not immediately obvious what that should mean, but MATLAB's interpretation is that there aren't any numbers that fit that description, so the result is

```
>> 1:-1

ans = Empty matrix: 1-by-0
```

If the matrix is empty, you might expect it to be "0-by-0," but there you have it. In any case, if you loop over an empty range, the loop never runs at all, which is why in this example the value of `total` is zero for any negative value of `n`.

If you are sure that you will never make a mistake, and that the preconditions of your functions will always be satisfied, then you don't have to check. But for the rest of us, it is dangerous to write a script, like this one, that quietly produces the wrong answer (or at least a meaningless answer) if the input value is negative. A better alternative is to use an if statement.

4.2 if

The if statement allows you to check for certain conditions and execute statements if the conditions are met. In the previous example, we could write:

```
if n<0
    ans = NaN
end
```

The syntax is similar to a for loop. The first line specifies the condition we are interested in; in this case we are asking if n is negative. If it is, MATLAB executes the body of the statement, which is the indented sequence of statements between the if and the end.

MATLAB doesn't require you to indent the body of an if statement, but it makes your code more readable, so you should do it, and don't make me tell you again.

In this example, the "right" thing to do if n is negative is to set ans = NaN, which is a standard way to indicate that the result is undefined (not a number).

If the condition is not satisfied, the statements in the body are not executed. Sometimes there are alternative statements to execute when the condition is false. In that case you can extend the if statement with an else clause.

The complete version of the previous example might look like this:

```
if n<0
    ans = NaN
else
    A1 = 1;
    total = 0;
    for i=1:n
        a = A1 * 0.5^(i-1);
        total = total + a;
    end
    ans = total
end
```

Statements like if and for that contain other statements are called **compound** statements. All compound statements end with, well, end.

In this example, one of the statements in the else clause is a for loop. Putting one compound statement inside another is legal and common, and sometimes called **nesting**.

4.3 Relational operators

The operators that compare values, like < and > are called **relational operators** because they test the relationship between two values. The result of a relational operator is one of the **logical values**: either 1, which represents “true,” or 0, which represents “false.”

Relational operators often appear in if statements, but you can also evaluate them at the prompt:

```
>> x = 5;
>> x < 10

ans = 1
```

You can assign a logical value to a variable:

```
>> flag = x > 10

flag = 0
```

A variable that contains a logical value is often called a **flag** because it flags the status of some condition.

The other relational operators are <= and >=, which are self-explanatory, ==, for “equal,” and ~=, for “not equal.” (In some logic notations, the tilde is the symbol for “not.”)

Don’t forget that == is the operator that tests equality, and = is the assignment operator. If you try to use = in an if statement, you get a syntax error:

```
if x=5
??? if x=5
        |
Error: The expression to the left of the equals sign is not a valid
target for an assignment.
```

MATLAB thinks you are making an assignment to a variable named if x!

4.4 Logical operators

To test if a number falls in an interval, you might be tempted to write something like 0 < x < 10, but that would be wrong, so very wrong. Unfortunately, in many cases, you will get the right answer for the wrong reason. For example:

```
>> x = 5;
>> 0 < x < 10            % right for the wrong reason

ans = 1
```

But don’t be fooled!

```
>> x = 17
>> 0 < x < 10             % just plain wrong

ans = 1
```

The problem is that MATLAB is evaluating the operators from left to right, so first it checks if `0<x`. It is, so the result is 1. Then it compares the logical value 1 (not the value of `x`) to 10. Since `1<10`, the result is true, even though `x` is not in the interval.

For beginning programmers, this is an evil, evil bug!

One way around this problem is to use a nested `if` statement to check the two conditions separately:

```
ans = 0
if 0<x
    if x<10
        ans = 1
    end
end
```

But it is more concise to use the AND operator, `&&`, to combine the conditions.

```
>> x = 5;
>> 0<x && x<10

ans = 1

>> x = 17;
>> 0<x && x<10

ans = 0
```

The result of AND is true if *both* of the operands are true. The OR operator, `||`, is true if *either or both* of the operands are true.

4.5 Vectors

The values we have seen so far are all single numbers, which are called **scalars** to contrast them with **vectors** and **matrices**, which are collections of numbers.

A vector in MATLAB is similar to a sequence in mathematics; it is a set of numbers that correspond to positive integers. What we called a “range” in the previous chapter was actually a vector.

In general, anything you can do with a scalar, you can also do with a vector. You can assign a vector value to a variable:

```
>> X = 1:5

X = 1      2      3      4      5
```

Variables that contain vectors are often capital letters. That's just a convention; MATLAB doesn't require it, but for beginning programmers it is a useful way to remember what is a scalar and what is a vector.

Just as with sequences, the numbers that make up the vector are called **elements**.

4.6 Vector arithmetic

You can perform arithmetic with vectors, too. If you add a scalar to a vector, MATLAB increments each element of the vector:

```
>> Y = X+5

Y = 6      7      8      9     10
```

The result is a new vector; the original value of X is not changed.

If you add two vectors, MATLAB adds the corresponding elements of each vector and creates a new vector that contains the sums:

```
>> Z = X+Y

Z = 7      9     11     13     15
```

But adding vectors only works if the operands are the same size. Otherwise:

```
>> W = 1:3

W = 1      2      3

>> X+W
??? Error using ==> plus
Matrix dimensions must agree.
```

The error message in this case is confusing, because we are thinking of these values as vectors, not matrices. The problem is a slight mismatch between math vocabulary and MATLAB vocabulary.

4.7 Everything is a matrix

In math (specifically in linear algebra) a vector is a one-dimensional sequence of values and a matrix is two-dimensional (and, if you want to think of it that way, a scalar is zero-dimensional). In MATLAB, everything is a matrix.

You can see this if you use the whos command to display the variables in the workspace. whos is similar to who except that it also displays the size and type of each variable.

First I'll make one of each kind of value:

```
>> scalar = 5

scalar = 5

>> vector = 1:5

vector = 1     2     3     4     5

>> matrix = ones(2,3)

matrix =

     1     1     1
     1     1     1
```

ones is a function that builds a new matrix with the given number of rows and columns, and sets all the elements to 1. Now let's see what we've got.

```
>> whos
  Name         Size                   Bytes  Class

  scalar       1x1                        8  double array
  vector       1x5                       40  double array
  matrix       2x3                       32  double array
```

According to MATLAB, everything is a double array: "double" is another name for double-precision floating-point numbers, and "array" is another name for a matrix.

The only difference is the size, which is specified by the number of rows and columns. The thing we called scalar is, according to MATLAB, a matrix with one row and one column. Our vector is really a matrix with one row and 5 columns. And, of course, matrix is a matrix.

The point of all this is that you can think of your values as scalars, vectors, and matrices, and I think you should, as long as you remember that MATLAB thinks everything is a matrix.

Here's another example where the error message only makes sense if you know what is happening under the hood:

```
>> X = 1:5

X = 1     2     3     4     5
```

```
>> Y = 1:5

Y = 1     2     3     4     5

>> Z = X*Y
??? Error using ==> mtimes
Inner matrix dimensions must agree.
```

First of all, mtimes is the MATLAB function that performs matrix multiplication. The reason the "inner matrix dimensions must agree" is that the way matrix multiplication is defined in linear algebra, the number of rows in X has to equal the number of columns in Y (those are the inner dimensions).

If you don't know linear algebra, this doesn't make much sense. When you saw X*Y you probably expected it to multiply each the the elements of X by the corresponding element of Y and put the results into a new vector. That operation is called **elementwise** multiplication, and the operator that performs it is .*:

```
>> X .* Y

ans = 1     4     9    16    25
```

We'll get back to the elementwise operators later; you can forget about them for now.

4.8 Indices

You can select elements of a vector with parentheses:

```
>> Y = 6:10

Y = 6     7     8     9    10

>> Y(1)

ans = 6

>> Y(5)

ans = 10
```

This means that the first element of Y is 6 and the fifth element is 10. The number in parentheses is called the **index** because it indicates which element of the vector you want.

The index can be any kind of expression.

```
>> i = 1;
>> Y(i+1)

ans = 7
```

Loops and vectors go together like the storm and rain. For example, this loop displays the elements of Y.

```
for i=1:5
     Y(i)
end
```

Each time through the loop we use a different value of i as an index into Y.

A limitation of this example is that we had to know the number of elements in Y. We can make it more general by using the length function, which returns the number of elements in a vector:

```
for i=1:length(Y)
     Y(i)
end
```

There. Now that will work for a vector of any length.

4.9 Indexing errors

An index can be any kind of expression, but the value of the expression has to be a positive integer, and it has to be less than or equal to the length of the vector. If it's zero or negative, you get this:

```
>> Y(0)
??? Subscript indices must either be real positive integers or
logicals.
```

"Subscript indices" is MATLAB's longfangled way to say "indices." "Real positive integers" means that complex numbers are out. And you can forget about "logicals" for now.

If the index is too big, you get this:

```
>> Y(6)
??? Index exceeds matrix dimensions.
```

There's the "m" word again, but other than that, this message is pretty clear.

Finally, don't forget that the index has to be an integer:

```
>> Y(1.5)
??? Subscript indices must either be real positive integers or
logicals.
```

4.10 Vectors and sequences

Vectors and sequences go together like ice cream and apple pie. For example, another way to evaluate the Fibonacci sequence is by storing successive values in a vector. Again, the definition of the Fibonacci sequence is $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 3$. In MATLAB, that looks like

```
F(1) = 1
F(2) = 1
for i=3:n
    F(i) = F(i-1) + F(i-2)
end
ans = F(n)
```

Notice that I am using a capital letter for the vector `F` and lower-case letters for the scalars `i` and `n`. At the end, the script extracts the final element of `F` and stores it in `ans`, since the result of this script is supposed to be the nth Fibonacci number, not the whole sequence.

If you had any trouble with Exercise 3.5, you have to appreciate the simplicity of this version. The MATLAB syntax is similar to the math notation, which makes it easier to check correctness. The only drawbacks are

- You have to be careful with the range of the loop. In this version, the loop runs from `3` to `n`, and each time we assign a value to the `i`th element. It would also work to "shift" the index over by two, running the loop from 1 to `n-2`:

  ```
  F(1) = 1
  F(2) = 1
  for i=1:n-2
      F(i+2) = F(i+1) + F(i)
  end
  ans = F(n)
  ```

  Either version is fine, but you have to choose one approach and be consistent. If you combine elements of both, you will get confused. I prefer the version that has `F(i)` on the left side of the assignment, so that each time through the loop it assigns the `i`th element.

- If you really only want the nth Fibonacci number, then storing the whole sequence wastes some storage space. But if wasting space makes your code easier to write and debug, that's probably ok.

Exercise 4.1 *Write a loop that computes the first* `n` *elements of the geometric sequence $A_{i+1} = A_i/2$ with $A_1 = 1$. Notice that the math notation puts A_{i+1} on the left side of the equality. When you translate to MATLAB, you may want to shift the index.*

4.11 Plotting vectors

Plotting and vectors go together like the moon and June, whatever that means. If you call `plot` with a single vector as an argument, MATLAB plots the indices on the x-axis and the elements on the y-axis. To plot the Fibonacci numbers we computed in the previous section:

```
plot(F)
```

This display is often useful for debugging, especially if your vectors are big enough that displaying the elements on the screen is unwieldy.

If you call `plot` with two vectors as arguments, MATLAB plots the second one as a function of the first; that is, it treats the first vector as a sequence of x values and the second as corresponding y value and plots a sequence of (x, y) points.

```
X = 1:5
Y = 6:10
plot(X, Y)
```

By default, MATLAB draws a blue line, but you can override that setting with the same kind of string we saw in Section 3.5. For example, the string `'ro-'` tells MATLAB to plot a red circle at each data point; the hyphen means the points should be connected with a line.

In this example, I stuck with the convention of naming the first argument `X` (since it is plotted on the x-axis) and the second `Y`. There is nothing special about these names; you could just as well plot `X` as a function of `Y`. MATLAB always treats the first vector as the “independent” variable, and the second as the “dependent” variable (if those terms are familiar to you).

4.12 Reduce

A frequent use of loops is to run through the elements of an array and add them up, or multiply them together, or compute the sum of their squares, etc. This kind of operation is called **reduce**, because it reduces a vector with multiple elements down to a single scalar.

For example, this loop adds up the elements of a vector named `X` (which we assume has been defined).

```
total = 0
for i=1:length(X)
    total = total + X(i)
end
ans = total
```

The use of `total` as an accumulator is similar to what we saw in Section 3.7. Again, we use the `length` function to find the upper bound of the range, so this loop will work regardless of the length of X. Each time through the loop, we add in the ith element of X, so at the end of the loop total contains the sum of the elements.

Exercise 4.2 *Write a similar loop that multiplies all the elements of a vector together. You might want to call the accumulator* product*, and you might want to think about the initial value you give it before the loop.*

4.13 Apply

Another common use of a loop is to run through the elements of a vector, perform some operation on the elements, and create a new vector with the results. This kind of operation is called **apply**, because you apply the operation to each element in the vector.

For example, the following loop computes a vector Y that contains the squares of the elements of X (assuming, again, that X is already defined).

```
for i=1:length(X)
    Y(i) = X(i)^2
end
```

Exercise 4.3 *Write a loop that computes a vector* Y *that contains the sines of the elements of* X*. To test your loop, write a script that*

1. *Uses* linspace *(see the documentation) to assign to* X *a vector with 100 elements running from 0 to* 2π*.*

2. *Uses your loop to store the sines in* Y*.*

3. *Plots the elements of* Y *as a function of the elements of* X*.*

4.14 Search

Yet another use of loops is to search the elements of a vector and return the index of the value you are looking for (or the first value that has a particular property). For example, if a vector contains the computed altitude of a falling object, you might want to know the index where the object touches down (assuming that the ground is at altitude 0).

To create some fake data, we'll use an extended version of the colon operator:

```
X = 10:-1:-10
```

The values in this range run from 10 to -10, with a **step size** of -1. The step size is the interval between elements of the range.

The following loop finds the index of the element 0 in X:

```
for i=1:length(X)
    if X(i) == 0
        ans = i
    end
end
```

One funny thing about this loop is that it keeps going after it finds what it is looking for. That might be what you want; if the target value appears more than one, this loop provides the index of the *last* one.

But if you want the index of the first one (or you know that there is only one), you can save some unnecessary looping by using the `break` statement.

```
for i=1:length(X)
    if X(i) == 0
        ans = i
        break
    end
end
```

`break` does pretty much what it sounds like. It ends the loop and proceeds immediately to the next statement after the loop (in this case, there isn't one, so the script ends).

This example demonstrates the basic idea of a search, but it also demonstrates a dangerous use of the `if` statement. Remember that floating-point values are often only approximately right. That means that if you look for a perfect match, you might not find it. For example, try this:

```
X = linspace(1,2)
for i=1:length(X)
    Y(i) = sin(X(i))
end
plot(X, Y)
```

You can see in the plot that the value of $\sin x$ goes through 0.9 in this range, but if you search for the index where `Y(i) == 0.9`, you will come up empty.

```
for i=1:length(Y)
    if Y(i) == 0.9
        ans = i
        break
    end
end
```

The condition is never true, so the body of the `if` statement is never executed.

Even though the plot shows a continuous line, don't forget that `X` and `Y` are sequences of discrete (and usually approximate) values. As a rule, you should (almost) never use the `==` operator to compare floating-point values. There are a number of ways to get around this limitation; we will get to them later.

Exercise 4.4 *Write a loop that finds the index of the first negative number in a vector and stores it in* `ans`*. If there are no negative numbers, it should set* `ans` *to -1 (which is not a legal index, so it is a good way to indicate the special case).*

4.15 Spoiling the fun

Experienced MATLAB programmers would never write the kind of loops in this chapter, because MATLAB provides simpler and faster ways to perform many reduce, filter and search operations.

For example, the `sum` function computes the sum of the elements in a vector and `prod` computes the product.

Many apply operations can be done with elementwise operators. The following statement is more concise than the loop in Section 4.13

```
Y = X .^ 2
```

Also, most built-in MATLAB functions work with vectors:

```
X = linspace(0, 2*pi)
Y = sin(X)
plot(X, Y)
```

Finally, the `find` function can perform search operations, but understanding it requires a couple of concepts we haven't got to, so for now you are better off on your own.

I started with simple loops because I wanted to demonstrate the basic concepts and give you a chance to practice. At some point you will probably have to write a loop for which there is no MATLAB shortcut, but you have to work your way up from somewhere.

If you understand loops and you are are comfortable with the shortcuts, feel free to use them! Otherwise, you can always write out the loop.

Exercise 4.5 *Write an expression that computes the sum of the squares of the elements of a vector.*

4.16 Glossary

compound: A statement, like `if` and `for`, that contains other statements in an indented body.

nesting: Putting one compound statement in the body of another.

relational operator: An operator that compares two values and generates a logical value as a result.

logical value: A value that represents either "true" or "false". MATLAB uses the values 1 and 0, respectively.

flag: A variable that contains a logical value, often used to store the status of some condition.

scalar: A single value.

vector: A sequence of values.

matrix: A two-dimensional collection of values (also called "array" in some MATLAB documentation).

index: An integer value used to indicate one of the values in a vector or matrix (also called subscript in some MATLAB documentation).

element: One of the values in a vector or matrix.

elementwise: An operation that acts on the individual elements of a vector or matrix (unlike some linear algebra operations).

reduce: A way of processing the elements of a vector and generating a single value; for example, the sum of the elements.

apply: A way of processing a vector by performing some operation on each of the elements, producing a vector that contains the results.

search: A way of processing a vector by examining the elements in order until one is found that has the desired property.

4.17 Exercises

Exercise 4.6 *The ratio of consecutive Fibonacci numbers, F_{n+1}/F_n, converges to a constant value as n increases. Write a script that computes a vector with the first n elements of a Fibonacci sequence (assuming that the variable* `n` *is defined), and then computes a new vector that contains the ratios of consecutive Fibonacci numbers. Plot this vector to see if it seems to converge. What value does it converge on?*

Exercise 4.7 *A certain famous system of differential equations can be approximated by a system of difference equations that looks like this:*

$$x_{i+1} = x_i + \sigma\,(y_i - x_i)\,dt \qquad (4.1)$$

$$y_{i+1} = y_i + [x_i(r - z_i) - y_i]\,dt \qquad (4.2)$$

$$z_{i+1} = z_i + (x_i y_i - b z_i)\,dt \qquad (4.3)$$

- *Write a script that computes the first 10 elements of the sequences X, Y and Z and stores them in vectors named* `X`*,* `Y` *and* `Z`*.*

  Use the initial values $X_1 = 1$, $Y_1 = 2$ and $Z_1 = 3$, with values $\sigma = 10$, $b = 8/3$ and $r = 28$, and with time step $dt = 0.01$.

- *Read the documentation for* `plot3` *and* `comet3` *and plot the results in 3 dimensions.*

- *Once the code is working, use semi-colons to suppress the output and then run the program with sequence length 100, 1000 and 10000.*

- *Run the program again with different starting conditions. What effect does it have on the result?*

- *Run the program with different values for σ, b and r and see if you can get a sense of how each variable affects the system.*

Exercise 4.8 *The logistic map is often cited as an example of how complex, chaotic behaviour can arise from simple non-linear dynamical equations [some of this description is adapted from the Wikipedia page on the logistic map]. It was popularized in a seminal 1976 paper by the biologist Robert May.*

It has been used to model the biomass of a species in the presence of limiting factors such as food supply and disease. In this case, there are two processes at work: (1) A reproductive process increases the biomass of the species in proportion to the current population. (2) A starvation process causes the biomass to decrease at a rate proportional to the carrying capacity of the environment less the current population.

Mathematically this can be written as

$$X_{i+1} = rX_i(1 - X_i)$$

where X_i is a number between zero and one that represents the biomass at year i, and r is a positive number that represents a combined rate for reproduction and starvation.

- *Write a script named* `logmap` *that computes the first 50 elements of X with* `r=3.9` *and* `X1=0.5`*, where* `r` *is the parameter of the logistic map and* `X1` *is the initial population.*

- *Plot the results for a range of values of r from 2.4 to 4.0. How does the behavior of the system change as you vary r.*

- *One way to characterize the effect of r is to make a plot with r on the x-axis and biomass on the y axis, and to show, for each value of r, the values of biomass that occur in steady state. See if you can figure out how to generate this plot.*

Chapter 5

Functions

5.1 Name Collisions

Remember that all of your scripts run in the same workspace, so if one script changes the value of a variable, all your other scripts see the change. With a small number of simple scripts, that's not a problem, but eventually the interactions between scripts become unmanageable.

For example, the following (increasingly familiar) script computes the sum of the first n terms in a geometric sequence, but it also has the **side-effect** of assigning values to A1, total, i and a.

```
A1 = 1;
total = 0;
for i=1:10
    a = A1 * 0.5^(i-1);
    total = total + a;
end
ans = total
```

If you were using any of those variable names before calling this script, you might be surprised to find, after running the script, that their values had changed. If you have two scripts that use the same variable names, you might find that they work separately and then break when you try to combine them. This kind of interaction is called a **name collision**.

As the number of scripts you write increases, and they get longer and more complex, name collisions become more of a problem. Avoiding this problem is one of the motivations for functions.

5.2 Functions

A **function** is like a script, except

- Each function has its own workspace, so any variables defined inside a function only exist while the function is running, and don't interfere with variables in other workspaces, even if they have the same name.
- Function inputs and outputs are defined carefully to avoid unexpected interactions.

To define a new function, you create an M-file with the name you want, and put a function definition in it. For example, to create a function named `myfunc`, create an M-file named `myfunc.m` and put the following definition into it.

```
function res = myfunc(x)
    s = sin(x)
    c = cos(x)
    res = abs(s) + abs(c)
end
```

The first word of the file has to be the word `function`, because that's how MATLAB tells the difference between a script and a function file.

A function definition is a compound statement. The first line is called the **signature** of the function; it defines the inputs and outputs of the function. In this case the **input variable** is named `x`. When this function is called, the argument provided by the user will be assigned to `x`.

The **output variable** is named `res`, which is short for "result." You can call the output variable whatever you want, but as a convention, I like to call it `res`. Usually the last thing a function does is assign a value to the output variable.

Once you have defined a new function, you call it the same way you call built-in MATLAB functions. If you call the function as a statement, MATLAB puts the result into `ans`:

```
>> myfunc(1)

s = 0.84147098480790

c = 0.54030230586814

res = 1.38177329067604

ans = 1.38177329067604
```

But it is more common (and better style) to assign the result to a variable:

```
>> y = myfunc(1)
```

```
s = 0.84147098480790

c = 0.54030230586814

res = 1.38177329067604

y = 1.38177329067604
```

While you are debugging a new function, you might want to display intermediate results like this, but once it is working, you will want to add semi-colons to make it a **silent function**. Most built-in functions are silent; they compute a result, but they don't display anything (except sometimes warning messages).

Each function has its own workspace, which is created when the function starts and destroyed when the function ends. If you try to access (read or write) the variables defined inside a function, you will find that they don't exist.

```
>> clear
>> y = myfunc(1);
>> who

Your variables are: y

>> s
??? Undefined function or variable 's'.
```

The only value from the function that you can access is the result, which in this case is assigned to y.

If you have variables named s or c in your workspace before you call myfunc, they will still be there when the function completes.

```
>> s = 1;
>> c = 1;
>> y = myfunc(1);
>> s, c

s = 1
c = 1
```

So inside a function you can use whatever variable names you want without worrying about collisions.

5.3 Documentation

At the beginning of every function file, you should include a comment that explains what the function does.

```
% res = myfunc (x)
% Compute the Manhattan distance from the origin to the
% point on the unit circle with angle (x) in radians.

function res = myfunc (x)
    s = sin(x);
    c = cos(x);
    res = abs(s) + abs(c);
end
```

When you ask for `help`, MATLAB prints the comment you provide.

```
>> help myfunc
  res = myfunc (x)
  Compute the Manhattan distance from the origin to the
  point on the unit circle with angle (x) in radians.
```

There are lots of conventions about what should be included in these comments. Among other things, it is a good idea to include

- The signature of the function, which includes the name of the function, the input variable(s) and the output variable(s).
- A clear, concise, abstract description of what the function does. An **abstract** description is one that leaves out the details of *how* the function works, and includes only information that someone using the function needs to know. You can put additional comments inside the function that explain the details.
- An explanation of what the input variables mean; for example, in this case it is important to note that `x` is considered to be an angle in radians.
- Any preconditions and postconditions.

5.4 Function names

There are three "gotchas" that come up when you start naming functions. The first is that the "real" name of your function is determined by the file name, *not* by the name you put in the function signature. As a matter of style, you should make sure that they are always the same, but if you make a mistake, or if you change the name of a function, it is easy to get confused.

In the spirit of making errors on purpose, change the name of the function in `myfunc` to `something_else`, and then run it again.

If this is what you put in `myfunc.m`:

```
function res = something_else (x)
    s = sin(x);
    c = cos(x);
    res = abs(s) + abs(c);
end
```

Then here's what you'll get:

```
>> y = myfunc(1);
>> y = something_else(1);
??? Undefined command/function 'something_else'.
```

The second gotcha is that the name of the file can't have spaces. For example, if you write a function and name the file `my func.m`, which the MATLAB editor will happily allow you to do, and then try to run it, you get:

```
>> y = my func(1)
??? y = my func(1)
           |
Error: Unexpected MATLAB expression.
```

The third gotcha is that your function names can collide with built-in MATLAB functions. For example, if you create an M-file named `sum.m`, and then call `sum`, MATLAB might call *your* new function, not the built-in version! Which one actually gets called depends on the order of the directories in the search path, and (in some cases) on the arguments. As an example, put the following code in a file named `sum.m`:

```
function res = sum(x)
   res = 7;
end
```

And then try this:

```
>> sum(1:3)

ans = 6

>> sum

ans = 7
```

In the first case MATLAB used the built-in function; in the second case it ran your function! This kind of interaction can be very confusing. Before you create a new function, check to see if there is already a MATLAB function with the same name. If there is, choose another name!

5.5 Multiple input variables

Functions can, and often do, take more than one input variable. For example, the following function takes two input variables, `a` and `b`:

```
function res = hypotenuse(a, b)
    res = sqrt(a^2 + b^2);
end
```

If you remember the Pythagorean Theorem, you probably figured out that this function computes the length of the hypotenuse of a right triangle if the lengths of the adjacent sides are `a` and `b`. (There is a MATLAB function called `hypot` that does the same thing.)

If we call it from the Command Window with arguments 3 and 4, we can confirm that the length of the third side is 5.

```
>> c = hypotenuse(3, 4)

c = 5
```

The arguments you provide are assigned to the input variables in order, so in this case 3 is assigned to `a` and 4 is assigned to `b`. MATLAB checks that you provide the right number of arguments; if you provide too few, you get

```
>> c = hypotenuse(3)
??? Input argument "b" is undefined.

Error in ==> hypotenuse at 2
    res = sqrt(a^2 + b^2);
```

This error message is confusing, because it suggests that the problem is in `hypotenuse` rather than in the function call. Keep that in mind when you are debugging.

If you provide too many arguments, you get

```
>> c = hypotenuse(3, 4, 5)
??? Error using ==> hypotenuse
Too many input arguments.
```

Which is a better message.

5.6 Logical functions

In Section 4.4 we used logical operators to compare values. MATLAB also provides **logical functions** that check for certain conditions and return logical values: 1 for “true” and 0 for “false”.

For example, `isprime` checks to see whether a number is prime.

```
>> isprime(17)

ans = 1

>> isprime(21)

ans = 0
```

The functions isscalar and isvector check whether a value is a scalar or vector; if both are false, you can assume it is a matrix (at least for now).

To check whether a value you have computed is an integer, you might be tempted to use isinteger. But that would be wrong, so very wrong. isinteger checks whether a value belongs to one of the integer types (a topic we have not discussed); it doesn't check whether a floating-point value happens to be integral.

```
>> c = hypotenuse(3, 4)

c = 5

>> isinteger(c)

ans = 0
```

To do that, we have to write our own logical function, which we'll call isintegral:

```
function res = isintegral(x)
    if round(x) == x
        res = 1;
    else
        res = 0;
    end
end
```

This function is good enough for most applications, but remember that floating-point values are only approximately right; in some cases the approximation is an integer but the actual value is not.

5.7 An incremental development example

Let's say that we want to write a program to search for "Pythagorean triples:" sets of integral values, like 3, 4 and 5, that are the lengths of the sides of a right triangle. In other words, we would like to find integral values a, b and c such that $a^2 + b^2 = c^2$.

Here are the steps we will follow to develop the program incrementally.

- Write a script named find_triples and start with a simple statement like x=5.
- Write a loop that enumerates values of a from 1 to 3, and displays them.
- Write a nested loop that enumerates values of b from 1 to 4, and displays them.
- Inside the loop, call hypotenuse to compute c and display it.

- Use `isintegral` to check whether c is an integral value.
- Use an if statement to print only the triples a, b and c that pass the test.
- Transform the script into a function.
- Generalize the function to take input variables that specify the range to search.

So the first draft of this program is `x=5`, which might seem silly, but if you start simple and add a little bit at a time, you will avoid a lot of debugging.

Here's the second draft:

```
for a=1:3
    a
end
```

At each step, the program is testable: it produces output (or another visible effect) that you can check.

5.8 Nested loops

The third draft contains a nested loop:

```
for a=1:3
    a
    for b=1:4
        b
    end
end
```

The inner loop gets executed 3 times, once for each value of `a`, so here's what the output loops like (I adjusted the spacing to make the structure clear):

```
>> find_triples

a = 1   b = 1
        b = 2
        b = 3
        b = 4

a = 2   b = 1
        b = 2
        b = 3
        b = 4

a = 3   b = 1
        b = 2
        b = 3
        b = 4
```

The next step is to compute c for each pair of values a and b.

```
for a=1:3
    for b=1:4
        c = hypotenuse(a, b);
        [a, b, c]
    end
end
```

To display the values of `a`, `b` and `c`, I am using a feature we haven't seen before. The bracket operator creates a new matrix which, when it is displayed, shows the three values on one line:

```
>> find_triples

ans = 1.0000    1.0000    1.4142
ans = 1.0000    2.0000    2.2361
ans = 1.0000    3.0000    3.1623
ans = 1.0000    4.0000    4.1231
ans = 2.0000    1.0000    2.2361
ans = 2.0000    2.0000    2.8284
ans = 2.0000    3.0000    3.6056
ans = 2.0000    4.0000    4.4721
ans = 3.0000    1.0000    3.1623
ans = 3.0000    2.0000    3.6056
ans = 3.0000    3.0000    4.2426
ans = 3         4         5
```

Sharp-eyed readers will notice that we are wasting some effort here. After checking $a = 1$ and $b = 2$, there is no point in checking $a = 2$ and $b = 1$. We can eliminate the extra work by adjusting the range of the second loop:

```
for a=1:3
    for b=a:4
        c = hypotenuse(a, b);
        [a, b, c]
    end
end
```

If you are following along, run this version to make sure it has the expected effect.

5.9 Conditions and flags

The next step is to check for integral values of c. This loop calls `isintegral` and prints the resulting logical value.

```
for a=1:3
    for b=a:4
```

```
        c = hypotenuse(a, b);
        flag = isintegral(c);
        [c, flag]
    end
end
```

By not displaying a and b I made it easy to scan the output to make sure that the values of c and flag look right.

```
>> find_triples

ans = 1.4142         0
ans = 2.2361         0
ans = 3.1623         0
ans = 4.1231         0
ans = 2.8284         0
ans = 3.6056         0
ans = 4.4721         0
ans = 4.2426         0
ans = 5              1
```

I chose the ranges for a and b to be small (so the amount of output is manageable), but to contain at least one Pythagorean triple. A constant challenge of debugging is to generate enough output to demonstrate that the code is working (or not) without being overwhelmed.

The next step is to use flag to display only the successful triples:

```
for a=1:3
    for b=a:4
        c = hypotenuse(a, b);
        flag = isintegral(c);
        if flag
            [a, b, c]
        end
    end
end
```

Now the output is elegant and simple:

```
>> find_triples

ans = 3     4     5
```

5.10 Encapsulation and generalization

As a script, this program has the side-effect of assigning values to a, b, c and flag, which would make it hard to use if any of those names were in use. By wrapping the code in a function, we can avoid name collisions; this process is called **encapsulation** because it isolates this program from the workspace.

In order to put the code we have written inside a function, we have to indent the whole thing. The MATLAB editor provides a shortcut for doing that, the Increase Indent command under the Text menu. Just don't forget to unselect the text before you start typing!

The first draft of the function takes no input variables:

```
function res = find_triples ()
    for a=1:3
        for b=a:4
            c = hypotenuse(a, b);
            flag = isintegral(c);
            if flag
                [a, b, c]
            end
        end
    end
end
```

The empty parentheses in the signature are not strictly necessary, but they make it apparent that there are no input variables. Similarly, when I call the new function, I like to use parentheses to remind me that it is a function, not a script:

```
>> find_triples()
```

The output variable isn't strictly necessary, either; it never gets assigned a value. But I put it there as a matter of habit, and also so my function signatures all have the same structure.

The next step is to generalize this function by adding input variables. The natural generalization is to replace the constant values 3 and 4 with a variable so we can search an arbitrarily large range of values.

```
function res = find_triples (n)
    for a=1:n
        for b=a:n
            c = hypotenuse(a, b);
            flag = isintegral(c);
            if flag
                [a, b, c]
            end
        end
    end
end
```

Here are the results for the range from 1 to 15:

```
>> find_triples(15)

ans = 3     4     5
ans = 5    12    13
ans = 6     8    10
ans = 8    15    17
ans = 9    12    15
```

Some of these are more interesting than others. The triples $5, 12, 13$ and $8, 15, 17$ are "new," but the others are just multiples of the $3, 4, 5$ triangle we already knew.

5.11 A misstep

When you change the signature of a function, you have to change all the places that call the function, too. For example, suppose I decided to add a third input variable to `hypotenuse`:

```
function res = hypotenuse(a, b, d)
    res = (a.^d + b.^d) ^ (1/d);
end
```

When `d` is 2, this does the same thing it did before. There is no practical reason to generalize the function in this way; it's just an example. Now when you run `find_triples`, you get:

```
>> find_triples(20)
??? Input argument "d" is undefined.

Error in ==> hypotenuse at 2
    res = (a.^d + b.^d) ^ (1/d);

Error in ==> find_triples at 7
            c = hypotenuse(a, b);
```

So that makes it pretty easy to find the error. This is an example of a development technique that is sometimes useful: rather than search the program for all the places that use `hypotenuse`, you can run the program and use the error messages to guide you.

But this technique is risky, especially if the error messages make suggestions about what to change. If you do what you're told, you might make the error message go away, but that doesn't mean the program will do the right thing. MATLAB doesn't know what the program is *supposed* to do, but you should.

And that brings us to the Eighth Theorem of debugging:

> Error messages sometimes tell you what's wrong, but they seldom tell you what to do (and when they try, they're usually wrong).

5.12 continue

As one final improvement, let's modify the function so that it only displays the "lowest" of each Pythagorean triple, and not the multiples.

The simplest way to eliminate the multiples is to check whether *a* and *b* share a common factor. If they do, then dividing both by the common factor yields a smaller, similar triangle that has already been checked.

MATLAB provides a `gcd` function that computes the greatest common divisor of two numbers. If the result is greater than 1, then *a* and *b* share a common factor and we can use the `continue` statement to skip to the next pair:

```
function res = find_triples (n)
    for a=1:n
        for b=a:n
            if gcd(a,b) > 1
                continue
            end
            c = hypotenuse(a, b);
            if isintegral(c)
                [a, b, c]
            end
        end
    end
end
```

`continue` causes the program to end the current iteration immediately (without executing the rest of the body), jump to the top of the loop, and "continue" with the next iteration.

In this case, since there are two loops, it might not be obvious which loop to jump to, but the rule is to jump to the inner-most loop (which is what we wanted).

I also simplified the program slightly by eliminating `flag` and using `isintegral` as the condition of the `if` statement.

Here are the results with `n=40`:

```
>> find_triples(40)

ans =  3     4     5
ans =  5    12    13
ans =  7    24    25
ans =  8    15    17
ans =  9    40    41
ans = 12    35    37
ans = 20    21    29
```

There is an interesting connection between Fibonacci numbers and Pythagorean triples. If F is a Fibonacci sequence, then

$$(F_n F_{n+3}, 2F_{n+1}F_{n+2}, F_{n+1}^2 + F_{n+2}^2)$$

is a Pythagorean triple for all $n \geq 1$.

Exercise 5.1 *Write a function named* `fib_triple` *that takes an input variable* `n`*, uses* `fibonacci2` *to compute the first* `n` *Fibonacci numbers, and then checks whether this formula produces a Pythagorean triple for each number in the sequence.*

5.13 Mechanism and leap of faith

Let's review the sequence of steps that occur when you call a function:

1. Before the function starts running, MATLAB creates a new workspace for it.
2. MATLAB evaluates each of the arguments and assigns the resulting values, in order, to the input variables (which live in the *new* workspace).
3. The body of the code executes. Somewhere in the body (often the last line) a value gets assigned to the output variable.
4. The function's workspace is destroyed; the only thing that remains is the value of the output variable and any side effects the function had (like displaying values or creating a figure).
5. The program resumes from where it left off. The value of the function call is the value of the output variable.

When you are reading a program and you come to a function call, there are two ways to interpret it:

- You can think about the mechanism I just described, and follow the execution of the program into the function and back, or
- You can take the "leap of faith": assume that the function works correctly, and go on to the next statement after the function call.

When you use built-in functions, it is natural to take the leap of faith, in part because you expect that most MATLAB functions work, and in part because you don't generally have access to the code in the body of the function.

But when you start writing your own functions, you will probably find yourself following the "flow of execution." This can be useful while you are learning, but as you gain experience, you should get more comfortable with the idea of writing a function, testing it to make sure it works, and then forgetting about the details of how it works.

Forgetting about details is called **abstraction**; in the context of functions, abstraction means forgetting about *how* a function works, and just assuming (after appropriate testing) that it works.

5.14 Glossary

side-effect: An effect, like modifying the workspace, that is not the primary purpose of a script.

name collision: The scenario where two scripts that use the same variable name interfere with each other.

input variable: A variable in a function that gets its value, when the function is called, from one of the arguments.

output variable: A variable in a function that is used to return a value from the function to the caller.

signature: The first line of a function definition, which specifies the names of the function, the input variables and the output variables.

silent function: A function that doesn't display anything or generate a figure, or have any other side-effects.

logical function: A function that returns a logical value (1 for "true" or 0 for "false").

encapsulation: The process of wrapping part of a program in a function in order to limit interactions (including name collisions) between the function and the rest of the program.

generalization: Making a function more versatile by replacing specific values with input variables.

abstraction: The process of ignoring the details of how a function works in order to focus on a simpler model of what the function does.

5.15 Exercises

Exercise 5.2 *Take any of the scripts you have written so far, encapsulate the code in an appropriately-named function, and generalize the function by adding one or more input variables.*

Make the function silent and then call it from the Command Window and confirm that you can display the output value.

Chapter 6

Zero-finding

6.1 Why functions?

The previous chapter explained some of the benefits of functions, including

- Each function has its own workspace, so using functions helps avoid name collisions.

- Functions lend themselves to incremental development: you can debug the body of the function first (as a script), then encapsulate it as a function, and then generalize it by adding input variables.

- Functions allow you to divide a large problem into small pieces, work on the pieces one at a time, and then assemble a complete solution.

- Once you have a function working, you can forget about the details of how it works and concentrate on what it does. This process of abstraction is an important tool for managing the complexity of large programs.

Another reason you should consider using functions is that many of the tools provided by MATLAB require you to write functions. For example, in this chapter we will use `fzero` to find solutions of nonlinear equations. Later we will use `ode45` to approximate solutions to differential equations.

6.2 Maps

In mathematics, a **map** is a correspondence between one set called the **range** and another set called the **domain**. For each element of the range, the map specifies the corresponding element of the domain.

You can think of a sequence as a map from positive integers to elements. You can think of a vector as a map from indices to elements. In these cases the maps are **discrete** because the elements of the range are countable.

You can also think of a function as a map from inputs to outputs, but in this case the range is **continuous** because the inputs can take any value, not just integers. (Strictly speaking, the set of floating-point numbers is discrete, but since floating-point numbers are meant to represent real numbers, we think of them as continuous.)

6.3 A note on notation

In this chapter I need to start talking about mathematical functions, and I am going to use a notation you might not have seen before.

If you have studied functions in a math class, you have probably seen something like

$$f(x) = x^2 - 2x - 3$$

which is supposed to mean that f is a function that maps from x to $x^2 - 2x - 3$. The problem is that $f(x)$ is also used to mean the value of f that corresponds to a particular value of x. So I don't like this notation. I prefer

$$f : x \rightarrow x^2 - 2x - 3$$

which means "f is the function that maps from x to $x^2 - 2x - 3$." In MATLAB, this would be expressed like this:

```
function res = error_func(x)
    res = x^2 - 2*x -3;
end
```

I'll explain soon why this function is called `error_func`. Now, back to our regularly-scheduled programming.

6.4 Nonlinear equations

What does it mean to "solve" an equation? That may seem like an obvious question, but I want to take a minute to think about it, starting with a simple example: let's say that we want to know the value of a variable, x, but all we know about it is the relationship $x^2 = a$.

If you have taken algebra, you probably know how to "solve" this equation: you take the square root of both sides and get $x = \sqrt{a}$. Then, with the satisfaction of a job well done, you move on to the next problem.

But what have you really done? The relationship you derived is equivalent to the relationship you started with—they contain the same information about x—so why is the second one preferable to the first?

There are two reasons. One is that the relationship is now "explicit in x;" because x is all alone on the left side, we can treat the right side as a recipe for computing x, assuming that we know the value of a.

The other reason is that the recipe is written in terms of operations we know how to perform. Assuming that we know how to compute square roots, we can compute the value of x for any value of a.

When people talk about solving an equation, what they usually mean is something like "finding an equivalent relationship that is explicit in one of the variables." In the context of this book, that's what I will call an **analytic solution**, to distinguish it from a **numerical solution**, which is what we are going to do next.

To demonstrate a numerical solution, consider the equation $x^2 - 2x = 3$. You could solve this analytically, either by factoring it or by using the quadratic equation, and you would discover that there are two solutions, $x = 3$ and $x = -1$. Alternatively, you could solve it numerically by rewriting it as $x = \sqrt{2x+3}$.

This equation is not explicit, since x appears on both sides, so it is not clear that this move did any good at all. But suppose that we had some reason to expect there to be a solution near 4. We could start with $x = 4$ as an "initial guess," and then use the equation $x = \sqrt{2x+3}$ iteratively to compute successive approximations of the solution.

Here's what would happen:

```
>> x = 4;

>> x = sqrt(2*x+3)

x = 3.3166

>> x = sqrt(2*x+3)

x = 3.1037

>> x = sqrt(2*x+3)

x = 3.0344

>> x = sqrt(2*x+3)

x = 3.0114
```

```
>> x = sqrt(2*x+3)

x = 3.0038
```

After each iteration, x is closer to the correct answer, and after 5 iterations, the relative error is about 0.1%, which is good enough for most purposes.

Techniques that generate numerical solutions are called **numerical methods**. The nice thing about the method I just demonstrated is that it is simple, but it doesn't always work as well as it did in this example, and it is not used very often in practice. We'll see one of the more practical alternatives in a minute.

6.5 Zero-finding

A nonlinear equation like $x^2 - 2x = 3$ is a statement of equality that is true for some values of x and false for others. A value that makes it true is a solution; any other value is a non-solution. But for any given non-solution, there is no sense of whether it is close or far from a solution, or where we might look to find one.

To address this limitation, it is useful to rewrite non-linear equations as zero-finding problems:

- The first step is to define an "error function" that computes how far a given value of x is from being a solution.

  In this example, the error function is

  $$f : x \rightarrow x^2 - 2x - 3$$

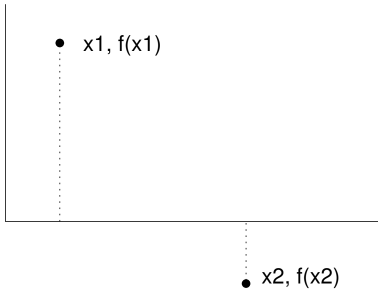
  Any value of x that makes $f(x) = 0$ is also a solution of the original equation.

- The next step is to find values of x that make $f(x) = 0$. These values are called **zeros of the function**, or sometimes roots.

Zero-finding lends itself to numerical solution because we can use the values of f, evaluated at various values of x, to make reasonable inferences about where to look for zeros.

For example, if we can find two values x_1 and x_2 such that $f(x_1) > 0$ and $f(x_2) < 0$, then we can be certain that there is at least one zero between x_1 and x_2 (provided that we know that f is continuous). In this case we would say that x_1 and x_2 bracket a zero.

Here's what this scenario might look like on a graph:



If this was all you knew about f, where would you go looking for a zero? If you said "halfway between x_1 and x_2," then congratulations! You just invented a numerical method called bisection!

If you said, "I would connect the dots with a straight line and compute the zero of the line," then congratulations! You just invented the secant method!

And if you said, "I would evaluate f at a third point, find the parabola that passes through all three points, and compute the zeros of the parabola," then... well, you probably didn't say that.

Finally, if you said, "I would use a built-in MATLAB function that combines the best features of several efficient and robust numerical methods," then you are ready to go on to the next section.

6.6 fzero

fzero is a built-in MATLAB function that combines the best features of several efficient and robust numerical methods.

In order to use fzero, you have to define a MATLAB function that computes the error function you derived from the original nonlinear equation, and you have to provide an initial guess at the location of a zero.

We've already seen an example of an error function:

```
function res = error_func(x)
    res = x^2 - 2*x -3;
end
```

You can call error_func from the Command Window, and confirm that there are zeros at 3 and -1.

```
>> error_func(3)
ans = 0

>> error_func(-1)
ans = 0
```

But let's pretend that we don't know exactly where the roots are; we only know that one of them is near 4. Then we could call `fzero` like this:

```
>> fzero(@error_func, 4)
ans = 3.0000
```

Success! We found one of the zeros.

The first argument is a **function handle** that names the M-file that evaluates the error function. The `@` symbol allows us to name the function without calling it. The interesting thing here is that you are not actually calling `error_func` directly; you are just telling `fzero` where it is. In turn, `fzero` calls your error function—more than once, in fact.

The second argument is the initial guess. If we provide a different initial guess, we get a different root (at least sometimes).

```
>> fzero(@error_func, -2)
ans = -1
```

Alternatively, if you know two values that bracket the root, you can provide both:

```
>> fzero(@error_func, [2,4])

ans = 3
```

The second argument here is actually a vector that contains two elements. The bracket operator is a convenient way (one of several) to create a new vector.

You might be curious to know how many times `fzero` calls your function, and where. If you modify `error_func` so that it displays the value of `x` every time it is called and then run `fzero` again, you get:

```
>> fzero(@error_func, [2,4])
x = 2
x = 4
x = 2.75000000000000
x = 3.03708133971292
x = 2.99755211623500
x = 2.99997750209270
x = 3.00000000025200
x = 3.00000000000000
x = 3
x = 3
ans = 3
```

Not surprisingly, it starts by computing $f(2)$ and $f(4)$. After each iteration, the interval that brackets the root gets smaller; `fzero` stops when the interval is so small that the estimated zero is correct to 16 digits. If you don't need that much precision, you can tell `fzero` to give you a quicker, dirtier answer (see the documentation for details).

6.7 What could go wrong?

The most common problem people have with fzero is leaving out the @. In that case, you get something like:

```
>> fzero(error_func, [2,4])
??? Input argument "x" is undefined.

Error in ==> error_func at 2
    x
```

Which is a very confusing error message. The problem is that MATLAB treats the first argument as a function call, so it calls error_func with no arguments. Since error_func requires one argument, the message indicates that the input argument is "undefined," although it might be clearer to say that you haven't provided a value for it.

Another common problem is writing an error function that never assigns a value to the output variable. In general, functions should *always* assign a value to the output variable, but MATLAB doesn't enforce this rule, so it is easy to forget. For example, if you write:

```
function res = error_func(x)
    y = x^2 - 2*x -3
end
```

and then call it from the Command Window:

```
>> error_func(4)

y = 5
```

It looks like it worked, but don't be fooled. This function assigns a value to y, and it displays the result, but when the function ends, y disappears along with the function's workspace. If you try to use it with fzero, you get

```
>> fzero(@error_func, [2,4])

y = -3

??? Error using ==> fzero
FZERO cannot continue because user supplied function_handle ==>
error_func failed with the error below.

Output argument "res" (and maybe others) not assigned during
call to "/home/downey/error_func.m (error_func)".
```

If you read it carefully, this is a pretty good error message (with the quibble that "output argument" is not a good synonym for "output variable").

You would have seen the same error message when you called error_func from the interpreter, if only you had assigned the result to a variable:

```
>> x = error_func(4)

y = 5

??? Output argument "res" (and maybe others) not assigned during
call to "/home/downey/error_func.m (error_func)".

Error in ==> error_func at 2
    y = x^2 - 2*x -3
```

You can avoid all of this if you remember these two rules:

- Functions should always assign values to their output variables[*].
- When you call a function, you should always do something with the result (either assign it to a variable or use it as part of an expression, etc.).

When you write your own functions and use them yourself, it is easy for mistakes to go undetected. But when you use your functions with MATLAB functions like `fzero`, you have to get it right!

Yet another thing that can go wrong: if you provide an interval for the initial guess and it doesn't actually contain a root, you get

```
>> fzero(@error_func, [0,1])

??? Error using ==> fzero
The function values at the interval endpoints must differ in sign.
```

There is one other thing that can go wrong when you use `fzero`, but this one is less likely to be your fault. It is possible that `fzero` won't be able to find a root.

`fzero` is generally pretty robust, so you may never have a problem, but you should remember that there is no guarantee that `fzero` will work, especially if you provide a single value as an initial guess. Even if you provide an interval that brackets a root, things can still go wrong if the error function is discontinuous.

6.8 Finding an initial guess

The better your initial guess (or interval) is, the more likely it is that `fzero` will work, and the fewer iterations it will need.

When you are solving problems in the real world, you will usually have some intuition about the answer. This intuition is often enough to provide a good initial guess for zero-finding.

[*] Well, ok, there are exceptions, including `find_triples`. Functions that don't return a value are sometimes called "commands," because they do something (like display values or generate a figure) but either don't have an output variable or don't make an assignment to it.

Another approach is to plot the function and see if you can approximate the zeros visually. If you have a function, like `error_func` that takes a scalar input variable and returns a scalar output variable, you can plot it with `ezplot`:

```
>> ezplot(@error_func, [-2,5])
```

The first argument is a function handle; the second is the interval you want to plot the function in.

By default `ezplot` calls your function 100 times (each time with a different value of `x`, of course). So you probably want to make your function silent before you plot it.

6.9 More name collisions

Functions and variables occupy the same "name-space," which means that whenever a name appears in an expression, MATLAB starts by looking for a variable with that name, and if there isn't one, it looks for a function.

As a result, if you have a variable with the same name as a function, the variable **shadows** the function. For example, if you assign a value to `sin`, and then try to use the `sin` function, you *might* get an error:

```
>> sin = 3;
>> x = 5;
>> sin(x)
??? Index exceeds matrix dimensions.
```

In this example, the problem is clear. Since the value of `sin` is a scalar, and a scalar is really a 1x1 matrix, MATLAB tries to access the 5th element of the matrix and finds that there isn't one. Of course, if there were more distance between the assignment and the "function call," this message would be pretty confusing.

But the only thing worse than getting an error message is *not* getting an error message. If the value of `sin` was a vector, or if the value of `x` was smaller, you would really be in trouble.

```
>> sin = 3;
>> sin(1)

ans = 3
```

Just to review, the sine of 1 is not 3!

The converse error can also happen if you try to access an undefined variable that also happens to be the name of a function. For example, if you have a function named `f`, and then try to increment a variable named `f` (and if you forget to initialize `f`), you get

```
>> f = f+1
??? Error:  "f" previously appeared to be used as a function or
command, conflicting with its use here as the name of a variable.
 A possible cause of this error is that you forgot to initialize the
 variable, or you have initialized it implicitly using load or eval.
```

At least, that's what you get if you are lucky. If this happens inside a function, MATLAB tries to call `f` as a function, and you get this

```
??? Input argument "x" is undefined.

Error in ==> f at 3
y = x^2 - a
```

There is no universal way to avoid these kind of collisions, but you can improve your chances by choosing variable names that don't shadow existing functions, and by choosing function names that you are unlikely to use as variables. That's why in Section 6.3 I called the error function `error_func` rather than `f`. I often give functions names that end in `func`, so that helps, too.

6.10 Debugging in four acts

When you are debugging a program, and especially if you are working on a hard bug, there are four things to try:

reading: Examine your code, read it back to yourself, and check that it means what you meant to say.

running: Experiment by making changes and running different versions. Often if you display the right thing at the right place in the program, the problem becomes obvious, but sometimes you have to spend some time to build scaffolding.

ruminating: Take some time to think! What kind of error is it: syntax, run-time, logical? What information can you get from the error messages, or from the output of the program? What kind of error could cause the problem you're seeing? What did you change last, before the problem appeared?

retreating: At some point, the best thing to do is back off, undoing recent changes, until you get back to a program that works, and that you understand. Then you can starting rebuilding.

Beginning programmers sometimes get stuck on one of these activities and forget the others. Each activity comes with its own failure mode.

For example, reading your code might help if the problem is a typographical error, but not if the problem is a conceptual misunderstanding. If you don't understand what your program does, you can read it 100 times and never see the error, because the error is in your head.

Running experiments can help, especially if you run small, simple tests. But if you run experiments without thinking or reading your code, you might fall into a pattern I call "random walk programming," which is the process of making random changes until the program does the right thing. Needless to say, random walk programming can take a long time.

The way out is to take more time to think. Debugging is like an experimental science. You should have at least one hypothesis about what the problem is. If there are two or more possibilities, try to think of a test that would eliminate one of them.

Taking a break sometimes helps with the thinking. So does talking. If you explain the problem to someone else (or even yourself), you will sometimes find the answer before you finish asking the question.

But even the best debugging techniques will fail if there are too many errors, or if the code you are trying to fix is too big and complicated. Sometimes the best option is to retreat, simplifying the program until you get to something that works, and then rebuild.

Beginning programmers are often reluctant to retreat, because they can't stand to delete a line of code (even if it's wrong). If it makes you feel better, copy your program into another file before you start stripping it down. Then you can paste the pieces back in a little bit at a time.

To summarize, here's the Ninth Theorem of debugging:

> Finding a hard bug requires reading, running, ruminating, and sometimes retreating. If you get stuck on one of these activities, try the others.

6.11 Glossary

analytic solution: A way of solving an equation by performing algebraic operations and deriving an explicit way to compute a value that is only known implicitly.

numerical solution: A way of solving an equation by finding a numerical value that satisfies the equation, often approximately.

numerical method: A method (or algorithm) for generating a numerical solution.

map: A correspondence between the elements of one set (the range) and the elements of another (the domain). You can think of sequences, vectors and functions as different kinds of maps.

range: The set of values a map maps from.

domain: The set of values a map maps to.

discrete set: A set, like the integers, whose elements are countable.

continuous set: A set, like the real numbers, whose elements are not countable. You can think of floating-point numbers as a continuous set.

zero (of a function): A value in the range of a function that maps to 0.

function handle: In MATLAB, a function handle is a way of referring to a function by name (and passing it as an argument) without calling it.

shadow: A kind of name collision in which a new definition causes an existing definition to become invisible. In MATLAB, variable names can shadow built-in functions (with hilarious results).

6.12 Exercises

Exercise 6.1 *1. Write a function called* `cheby6` *that evaluates the 6th Chebyshev polynomial. It should take an input variable, x, and return*

$$32x^6 - 48x^4 + 18x^2 - 1 \tag{6.1}$$

2. Use `ezplot` *to display a graph of this function in the interval from 0 to 1. Estimate the location of any zeros in this range.*

3. Use `fzero` *to find as many different roots as you can. Does* `fzero` *always find the root that is closest to the initial guess?*

Exercise 6.2 *The density of a duck, ρ, is $0.3 g/cm^3$ (0.3 times the density of water).*

The volume of a sphere[†] with radius r is $\frac{4}{3}\pi r^3$.

If a sphere with radius r is submerged in water to a depth d, the volume of the sphere below the water line is

$$volume = \frac{\pi}{3}(3rd^2 - d^3) \quad as\ long\ as \quad d < 2r$$

An object floats at the level where the weight of the displaced water equals the total weight of the object.

[†] This example is adapted from Gerald and Wheatley, *Applied Numerical Analysis*, Fourth Edition, Addison-Wesley, 1989.

Assuming that a duck is a sphere with radius 10 cm, at what depth does a duck float?

Here are some suggestions about how to proceed:

- *Write an equation relating ρ, d and r.*
- *Rearrange the equation so the right-hand side is zero. Our goal is to find values of d that are roots of this equation.*
- *Write a MATLAB function that evaluates this function. Test it, then make it a quiet function.*
- *Make a guess about the value of d_0 to use as a starting place.*
- *Use* `fzero` *to find a root near d_0.*
- *Check to make sure the result makes sense. In particular, check that $d < 2r$, because otherwise the volume equation doesn't work!*
- *Try different values of ρ and r and see if you get the effect you expect. What happens as ρ increases? Goes to infinity? Goes to zero? What happens as r increases? Goes to infinity? Goes to zero?*

Chapter 7

Functions of vectors

7.1 Functions and files

So far we have only put one function in each file. It is also possible to put more than one function in a file, but only the first one, the **top-level function** can be called from the Command Window. The other **helper functions** can be called from anywhere inside the file, but not from any other file.

Large programs almost always require more than one function; keeping all the functions in one file is convenient, but it makes debugging difficult because you can't call helper functions from the Command Window.

To help with this problem, I often use the top-level function to develop and test my helper functions. For example, to write a program for Exercise 6.2, I would create a file named `duck.m` and start with a top-level function named `duck` that takes no input variables and returns no output value.

Then I would write a function named `error_func` to evaluate the error function for `fzero`. To test `error_func` I would call it from `duck` and then call `duck` from the Command Window.

Here's what my first draft might look like:

```
function res = duck()
    error = error_func(10)
end

function res = error_func(h)
    rho = 0.3;      % density in g / cm^3
    r = 10;         % radius in cm
    res = h;
end
```

The line `res = h` isn't finished yet, but this is enough code to test. Once I finished and tested `error_func`, I would modify `duck` to use `fzero`.

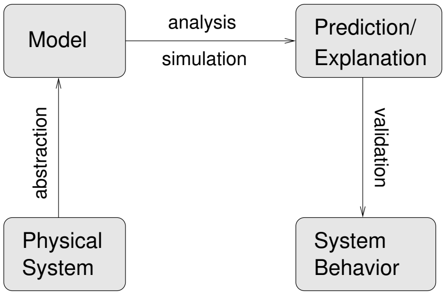
For this problem I might only need two functions, but if there were more, I could write and test them one at a time, and then combine them into a working program.

7.2 Physical modeling

Most of the examples so far have been about mathematics; Exercise 6.2, the "duck problem," is the first example we have seen of a physical system. If you didn't work on this exercise, you should at least go back and read it.

This book is supposed to be about **physical modeling**, so it might be a good idea to explain what that is. Physical modeling is a process for making predictions about physical systems and explaining their behavior. A **physical system** is something in the real world that we are interested in, like a duck.

The following figure shows the steps of this process:



A **model** is a simplified description of a physical system. The process of building a model is called **abstraction**. In this context, "abstract" is the opposite of "realistic;" an abstract model bears little direct resemblance to the physical system it models, in the same way that abstract art does not directly depict objects in the real world. A realistic model is one that includes more details and corresponds more directly to the real world.

Abstraction involves making justified decisions about which factors to include in the model and which factors can be simplified or ignored. For example, in the duck problem, we took into account the density of the duck and the buoyancy of water, but we ignored the buoyancy of the duck due to displacement of air and the dynamic effect of paddling feet. We also simplified the geometry of the duck by assuming that the underwater parts of a duck are similar to a segment of a sphere. And we used coarse estimates of the size and weight of the duck.

Some of these decisions are justifiable. The density of the duck is much higher than the density of air, so the effect of buoyancy in air is probably small. Other decisions, like the spherical geometry, are harder to justify, but very helpful. The actual geometry of a duck is complicated; the sphere model makes it possible to generate an approximate answer without making detailed measurements of real ducks.

A more realistic model is not necessarily better. Models are useful because they can be analyzed mathematically and simulated computationally. Models that are too realistic might be difficult to simulate and impossible to analyze.

A model is successful if it is good enough for its purpose. If we only need a rough idea of the fraction of a duck that lies below the surface, the sphere model is good enough. If we need a more precise answer (for some reason) we might need a more realistic model.

Checking whether a model is good enough is called **validation**. The strongest form of validation is to make a measurement of an actual physical system and compare it to the prediction of a model.

If that is infeasible, there are weaker forms of validation. One is to compare multiple models of the same system. If they are inconsistent, that is an indication that (at least) one of them is wrong, and the size of the discrepancy is a hint about the reliability of their predictions.

We have only seen one physical model so far, so parts of this discussion may not be clear yet. We will come back to these topics later, but first we should learn more about vectors.

7.3 Vectors as input variables

Since many of the built-in functions take vectors as arguments, it should come as no surprise that you can write functions that take vectors. Here's a simple (silly) example:

```
function res = display_vector(X)
    X
end
```

There's nothing special about this function at all. The only difference from the scalar functions we've seen is that I used a capital letter to remind me that `X` is a vector.

This is another example of a function that doesn't actually have a return value; it just displays the value of the input variable:

```
>> display_vector(1:3)

X = 1     2     3
```

Here's a more interesting example that encapsulates the code from Section 4.12 that adds up the elements of a vector:

```
function res = mysum(X)
    total = 0;
    for i=1:length(X)
        total = total + X(i);
    end
    res = total;
end
```

I called it `mysum` to avoid a collision with the built-in function `sum`, which does pretty much the same thing.

Here's how you call it from the Command Window:

```
>> total = mysum(1:3)

total = 6
```

Because this function has a return value, I made a point of assigning it to a variable.

7.4 Vectors as output variables

There's also nothing wrong with assigning a vector to an output variable. Here's an example that encapsulates the code from Section 4.13:

```
function res = myapply(X)
    for i=1:length(X)
        Y(i) = X(i)^2
    end
    res = Y
end
```

Ideally I would have changed the name of the output variable to `Res`, as a reminder that it is supposed to get a vector value, but I didn't.

Here's how `myapply` works:

```
>> V = myapply(1:3)

V = 1     4     9
```

Exercise 7.1 *Write a function named* `find_target` *that encapsulates the code, from Section 4.14, that finds the location of a target value in a vector.*

7.5 Vectorizing your functions

Functions that work on vectors will almost always work on scalars as well, because MATLAB considers a scalar to be a vector with length 1.

```
>> mysum(17)

ans = 17

>> myapply(9)

ans = 81
```

Unfortunately, the converse is not always true. If you write a function with scalar inputs in mind, it might not work on vectors.

But it might! If the operators and functions you use in the body of your function work on vectors, then your function will probably work on vectors.

For example, here is the very first function we wrote:

```
function res = myfunc (x)
    s = sin(x)
    c = cos(x)
    res = abs(s) + abs(c)
end
```

And lo! It turns out to work on vectors:

```
>> Y = myfunc(1:3)

Y = 1.3818    1.3254    1.1311
```

At this point, I want to take a minute to acknowledge that I have been a little harsh in my presentation of MATLAB, because there are a number of features that I think make life harder than it needs to be for beginners. But here, finally, we are seeing features that show MATLAB's strengths.

Some of the other functions we wrote don't work on vectors, but they can be patched up with just a little effort. For example, here's `hypotenuse` from Section 5.5:

```
function res = hypotenuse(a, b)
    res = sqrt(a^2 + b^2);
end
```

This doesn't work on vectors because the ^ operator tries to do matrix exponentiation, which only works on square matrices.

```
>> hypotenuse(1:3, 1:3)
??? Error using ==> mpower
Matrix must be square.
```

But if you replace ^ with the elementwise operator `.^`, it works!

```
>> A = [3,5,8];
>> B = [4,12,15];
>> C = hypotenuse(A, B)

C = 5    13    17
```

In this case, it matches up corresponding elements from the two input vectors, so the elements of C are the hypotenuses of the pairs $(3,4)$, $(5,12)$ and $(8,15)$, respectively.

In general, if you write a function using only elementwise operators and functions that work on vectors, then the new function will also work on vectors.

7.6 Sums and differences

Another common vector operation is **cumulative sum**, which takes a vector as an input and computes a new vector that contains all of the partial sums of the original. In math notation, if V is the original vector, then the elements of the cumulative sum, C, are:

$$C_i = \sum_{j=1}^{i} V_j$$

In other words, the ith element of C is the sum of the first i elements from V. MATLAB provides a function named cumsum that computes cumulative sums:

```
>> V = 1:5

V = 1     2     3     4     5

>> C = cumsum(V)

C = 1     3     6    10    15
```

Exercise 7.2 *Write a function named* cumulative_sum *that uses a loop to compute the cumulative sum of the input vector.*

The inverse operation of cumsum is diff, which computes the difference between successive elements of the input vector.

```
>> D = diff(C)

D = 2     3     4     5
```

Notice that the output vector is shorter by one than the input vector. As a result, MATLAB's version of diff is not exactly the inverse of cumsum. If it were, then we would expect cumsum(diff(X)) to be X:

```
>> cumsum(diff(V))

ans = 1     2     3     4
```

But it isn't.

Exercise 7.3 *Write a function named* `mydiff` *that computes the inverse of* `cumsum`*, so that* `cumsum(mydiff(X))` *and* `mydiff(cumsum(X))` *both return* `X`*.*

7.7 Products and ratios

The multiplicative version of `cumsum` is `cumprod`, which computes the **cumulative product**. In math notation, that's:

$$P_i = \prod_{j=1}^{i} V_j$$

In MATLAB, that looks like:

```
>> V = 1:5

V = 1     2     3     4     5

>> P = cumprod(V)

P = 1     2     6     24    120
```

Exercise 7.4 *Write a function named* `cumulative_prod` *that uses a loop to compute the cumulative product of the input vector.*

MATLAB doesn't provide the multiplicative version of `diff`, which would be called `ratio`, and which would compute the ratio of successive elements of the input vector.

Exercise 7.5 *Write a function named* `myratio` *that computes the inverse of* `cumprod`*, so that* `cumprod(myratio(X))` *and* `myratio(cumprod(X))` *both return* `X`*.*

You can use a loop, or if you want to be clever, you can take advantage of the fact that $e^{\ln a + \ln b} = ab$*.*

If you apply `myratio` *to a vector that contains Fibonacci numbers, you can confirm that the ratio of successive elements converges on the golden ratio,* $(1 + \sqrt{5})/2$ *(see Exercise 4.6).*

7.8 Existential quantification

It is often useful to check the elements of a vector to see if there are any that satisfy a condition. For example, you might want to know if there are any positive elements. In logic, this condition is called **existential quantification**, and it is denoted with the symbol ∃, which is pronounced "there exists." For example, this expression

$$\exists x \text{ in } S : x > 0$$

means, "there exists some element x in the set S such that $x > 0$." In MATLAB it is natural to express this idea with a logical function, like `exists`, that returns 1 if there is such an element and 0 if there is not.

```
function res = exists(X)
    for i=1:length(X)
        if X(i) > 0
            res = 1;
            return
        end
    end
    res = 0;
end
```

We haven't seen the `return` statement before; it is similar to `break` except that it breaks out of the whole function, not just the loop. That behavior is what we want here because as soon as we find a positive element, we know the answer (it exists!) and we can end the function immediately without looking at the rest of the elements.

If we exit at the end of the loop, that means we didn't find what we were looking for (because if we had, we would have hit the `return` statement).

7.9 Universal quantification

Another common operation on vectors is to check whether *all* of the elements satisfy a condition, which is known to logicians as **universal quantification** and denoted with the symbol ∀ which is pronounced "for all." So this expression

$$\forall x \text{ in } S : x > 0$$

means "for all elements, x, in the set S, $x > 0$."

A slightly silly way to evaluate this expression in MATLAB is to count the number of elements that satisfy the condition. A better way is to reduce the problem to existential quantification; that is, to rewrite

$$\forall x \text{ in } S : x > 0$$

as

$$\sim \exists x \text{ in } S : x \leq 0$$

Where $\sim \exists$ means "does not exist." In other words, checking that all the elements are positive is the same as checking that there are no elements that are non-positive.

Exercise 7.6 *Write a function named* `forall` *that takes a vector and returns 1 if all of the elements are positive and 0 if there are any non-positive elements.*

7.10 Logical vectors

When you apply a logical operator to a vector, the result is a **logical vector**; that is, a vector whose elements are the logical values 1 and 0.

```
>> V = -3:3

V = -3    -2    -1      0     1     2     3

>> L = V>0

L =  0     0     0     0     1     1     1
```

In this example, `L` is a logical vector whose elements correspond to the elements of `V`. For each positive element of `V`, the corresponding element of `L` is 1.

Logical vectors can be used like flags to store the state of a condition. They are also often used with the `find` function, which takes a logical vector and returns a vector that contains the indices of the elements that are "true."

Applying `find` to `L` yields

```
>> find(L)

ans = 5     6     7
```

which indicates that elements 5, 6 and 7 have the value 1.

If there are no "true" elements, the result is an empty vector.

```
>> find(V>10)

ans = Empty matrix: 1-by-0
```

This example computes the logical vector and passes it as an argument to `find` without assigning it to an intermediate variable. You can read this version abstractly as "find the indices of elements of `V` that are greater than 10."

We can also use `find` to write `exists` more concisely:

```
function res = exists(X)
    L = find(X>0)
    res = length(L) > 0
end
```

Exercise 7.7 *Write a version of* `forall` *using* `find`*.*

7.11 Glossary

top-level function: The first function in an M-file; it is the only function that can be called from the Command Window or from another file.

helper function: A function in an M-file that is not the top-level function; it only be called from another function in the same file.

physical modeling: A process for making predictions about physical systems and explaining their behavior.

physical system: Something in the real world that we are interested in studying.

model : A simplified description of a physical system that lends itself to analysis or simulation.

abstraction: The process of building a model by making decisions about what factors to simplify or ignore.

validation: Checking whether a model is adequate for its purpose.

existential quantification: A logical condition that expresses the idea that "there exists" an element of a set with a certain property.

universal quantification: A logical condition that expresses the idea that all elements of a set have a certain property.

logical vector: A vector, usually the result of applying a logical operator to a vector, that contains logical values 1 and 0.