Solution by Radicals of the Cubic: From Equations to Groups and from Real to Complex Numbers

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2.1 Introduction

Several ancient civilizations—the Babylonian, Egyptian, Chinese, and Indian—dealt with the solution of polynomial equations, mainly linear. The Babylonians (ca. 1600 BC ff.) were particularly proficient "algebraists". They were able to solve quadratic equations, as well as equations that *lead to* quadratics—for example, $x+y=a$ and $x^2+y^2=b$—by methods similar to ours. The equations were given in the form of "word problems", and were often expressed in geometric language. Here is a typical example [8, p. 24]:

> I summed the area and two-thirds of my square-side and it was 0;35 [35/60 in sexagesimal notation]. [What is the side of my square?]

In modern notation, the problem is to solve the equation $x^2+(2/3)x=35/60$. See [8, p. 24] for the Babylonians' solution of this equation.

The Chinese (ca. 200 BC ff.) and the Indians (ca. 600 BC ff.)—in each case the dates are very rough—made considerable advances in algebra. For example, both allowed negative coefficients in their equations—though not negative roots—and admitted two roots for a quadratic equation. They also described procedures for manipulating equations. The Chinese had methods for approximating roots of polynomial equations of any degree, and they solved *systems* of linear equations using "matrices" (rectangular arrays of numbers) well before such techniques were developed in Western Europe. The mathematics of the ancient Greeks, in particular their geometry and number theory, was relatively advanced, but their algebra was rather weak. (Note however that Diophantus (fl. ca. 250 AD), in his great number-theoretic work *Arithmetica,* introduced various algebraic symbols [1].) Book II of Euclid's remarkable work *Elements* (ca. 300 BC) presents, in geometric language, results which are familiar to *us* as algebraic, but most modern scholars believe that the Greeks of this period were *not* thinking algebraically.

Islamic mathematicians made important contributions in algebra between the ninth and fifteenth centuries. Among the foremost was Muhammad ibn-Mūsā al-Khwārizmī, dubbed by some "the Euclid of algebra" because he systematized the subject as it then existed and made it into an independent field of study. He did this in his book *al-jabr w al-muqabalah*. "Al-jabr", from which stems our word "algebra", denotes the moving of a negative term of an equation to the other side so as to make it positive, and "al-muqabalah" refers to cancelling equal (positive) terms on the two sides of an equation. These are of course basic procedures for solving polynomial equations. al-Khwārizmī, from whose name is derived the word "algorithm", applied these procedures to the solution of quadratic equations, which he classified into five types: $ax^2 = bx$, $ax^2 = b$, $ax^2 + bx = c$, $ax^2 + c = bx$, and $ax^2 = bx + c$. This categorization was necessary since al-Khwārizmī did not admit negative coefficients or zero into the number system. He also had no algebraic notation, so that his problems and solutions were expressed rhetorically (in words). He did however offer (geometric) justification for his solution procedures.

2.2 Cubic and Quartic Equations

The Babylonians (as we mentioned) were solving quadratic equations by about 1600 BC, using essentially an equivalent of our "quadratic formula". A natural question was therefore whether *cubic* equations could be solved using "similar" formulas; three thousand years would pass before the answer was discovered. It was a great event in algebra when mathematicians of the sixteenth century succeeded in solving—by radicals—not only cubic but also quartic equations. This accomplishment was very much in character with the mood of the Renaissance—which wanted not only to absorb the classic works of the ancients but to strike out in new directions. Indeed, the solution of the cubic unquestionably proved a far-reaching departure.

A "solution by radicals" of a polynomial equation is a formula giving the roots of the equation in terms of its coefficients. The only permissible operations to be applied to the coefficients are the four algebraic operations (addition, subtraction, multiplication, and division) and the extraction of roots (square roots, cube roots, and so on, that is, "radicals"). For example, the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is a solution by radicals of the equation $ax^2 + bx + c = 0$.

A solution by radicals of the cubic was first *published* in 1545 by Girolamo Cardano, in his *Ars Magna* (*The Great Art,* referring to algebra); it was *discovered* earlier by Scipione del Ferro and by Niccolò Tartaglia. The latter had passed on his method to Cardano, who had promised that he would not publish it; but he did. That is one version of events, which involved considerable drama and passion. A blow-by-blow account is given by Oysten Ore [13, pp. 53–107]. Here is Cardano's own rendition [8, p. 63]:

> Scipio Ferro of Bologna well-nigh thirty years ago [i.e., ca. 1515] discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Nicolò Tartaglia of Brescia gave Nicolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult.

What came to be known as "Cardano's formula" for the solution of the cubic $x^3 = ax + b$ is given by

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}.$$

Girolamo Cardano (1501–1576)

See for example [1, 2, 5]. Several comments are in order:

i. Cardano used essentially no symbols, so his “formula” giving the solution of the cubic was expressed rhetorically.
ii. He was usually content with determining a single root of a cubic. But in fact, if a proper choice is made of the cube roots involved, all three roots of the equation can be determined from his formula.
iii. The coefficients and roots of the cubics he considered were specific *positive numbers,* so that he viewed (say) $x^3 = ax + b$ and $x^3 + ax = b$ as distinct. He devoted a chapter to the solution of each, and gave *geometric* justifications [16, p. 63 ff.].
iv. Negative numbers are found occasionally in his work, but he mistrusted them, and called them “fictitious”. Irrational numbers were admitted as roots.

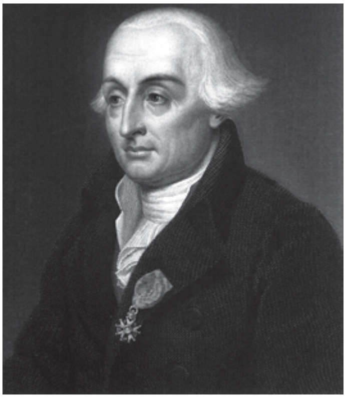
The solution by radicals of polynomial equations of the fourth degree—*quartics*—soon followed. The key idea was to reduce the solution of a quartic to that of a cubic. Ludovico Ferrari was the first to solve such equations, and his work was included in Cardano’s *The Great Art* [4].

It should be pointed out that cubic equations had arisen—in geometric guise—already in ancient Greece (ca. 400 BC), in connection with the problem of trisecting an angle, and that methods for finding *approximate* roots of cubics and quartics were known, for example by Chinese and Moslem mathematicians, well before such equations were solved by radicals. The latter solutions, though exact, were of little *practical* value. But the ramifications of these “impractical” ideas were very significant, and will now be briefly sketched.

2.3 Beyond the Quartic: Lagrange

Having solved the cubic and quartic by radicals, mathematicians turned to finding a solution by radicals of the quintic (degree-five polynomial)—a quest that would take nearly 300 years. Some of the most distinguished mathematicians of the seventeenth and eighteenth centuries, among them François Viète, René Descartes, Gottfried Wilhelm Leibniz, Leonhard Euler, and Étienne Bezout, tackled the problem. The strategy was to seek new approaches to the solutions of the cubic and quartic, in the hope that at least one of them would generalize to the quintic.

Joseph Louis Lagrange (1736–1830)



But to no avail: although new ideas for solving the cubic and quartic *were* found, they did not yield the desired extensions. One approach, however, undertaken by Joseph Louis Lagrange in a paper of 1770 entitled *Reflections on the Algebraic Solution of Equations*, proved promising. Lagrange analyzed the various methods devised by his predecessors for solving cubic and quartic equations, and saw that—since those methods did not work when applied to the quintic—a deeper scrutiny was required. In his own words [17, p. 127]:

> I propose in this memoir to examine the various methods found so far for the algebraic solution of equations, to reduce them to general principles, and to let us see *a priori* why these methods succeed for the third and fourth degree, and fail for higher degrees.

Here are some of the key ideas of Lagrange's approach. With each polynomial equation of arbitrary degree n he associated a "resolvent equation", as follows: let f(x) be the original equation, with roots $x_1, x_2, x_3, \ldots, x_n$. (As is the usual practice, we denote by "f(x)" both the polynomial and the polynomial equation.) Pick a rational function $R(x_1, x_2, x_3, \ldots, x_n)$ of the roots and coefficients of f(x). (Lagrange described a method for doing this.) Consider the different values which $R(x_1, x_2, x_3, \ldots, x_n)$ assumes under all the n! permutations of the roots $x_1, x_2, x_3, \ldots, x_n$ of f(x). If these values are denoted by $y_1, y_2, y_3, \ldots, y_k$, the "resolvent equation" is $(x-y_1)(x-y_2)\ldots(x-y_k)$. Lagrange showed that k divides n!—the source of what we call "Lagrange's theorem" in group theory.

For example, if f(x) is a quartic with roots x_1, x_2, x_3, x_4, then $R(x_1, x_2, x_3, x_4)$ may be taken to be $x_1x_2 + x_3x_4$, and this function assumes three distinct values under the 24 permutations of x_1, x_2, x_3, and x_4. Thus, the resolvent equation of a quartic is a cubic. However, in carrying over this analysis to the quintic, Lagrange found that the resolvent equation is of *degree six,* rather than the hoped-for degree four.

Although Lagrange did not succeed in settling the problem of the solvability of the quintic by radicals, his work was a milestone. It was the first time that an association was made between the solutions of a polynomial equation and the permutations of its roots. In fact, Lagrange speculated that the study of the permutations of the roots of an equation was the cornerstone of the theory of algebraic equations—"the genuine principles of the solution of equations", as he put it [17, p. 146]. He was of course vindicated in this by Evariste Galois.

Evariste Galois (1811–1832)

2.4 Ruffini, Abel, Galois

Paolo Ruffini and Niels-Henrik Abel proved (in 1799 and 1826, respectively) the unsolvability by radicals of the "general quintic". In fact, they proved the unsolvability of the "general equation" of degree n for *every* $n > 4$. They did this by building on Lagrange's pioneering ideas on resolvents. Lagrange had shown that a necessary condition for the solvability of the *general* polynomial equation of degree n is the existence of a resolvent of degree less than n. (A "general equation" is an equation with *arbitrary literal coefficients*.) Ruffini and Abel showed that such resolvents *do not exist* for *any* $n > 4$. (Abel proved this result without knowing of Ruffini's work; in any case, Ruffini's work had a significant gap.)

Although the *general* polynomial equation of degree 4 is unsolvable by radicals, some specific equations of this form *are* solvable; for example, $x^n - 1 = 0$ is solvable by radicals for every $n > 4$. Galois characterized those equations that are solvable by radicals in terms of *group theory*: A polynomial is solvable by radicals if and only if its "Galois group" is "solvable". To prove this result Galois founded the elements of permutation group theory and introduced in it various important concepts, such as Galois group, normal subgroup, and solvable group. Thus ended, in the early 1830s, the great saga—beginning with Cardano in 1545—of solvability by radicals of equations of degrees greater than 2.

2.5 Complex Numbers: Birth

A hugely important development arising from the solution of the cubic by radicals was the introduction of *complex numbers*.

Recall that Cardano's solution of the cubic $x^3 = ax + b$ is given by

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}.$$

Consider the cubic $x^3 = 9x + 2$. Its solution, using the above formula, is

$$x = \sqrt[3]{\tfrac{2}{2} + \sqrt{(\tfrac{2}{2})^2 - (\tfrac{9}{3})^3}} + \sqrt[3]{\tfrac{2}{2} - \sqrt{(\tfrac{2}{2})^2 - (\tfrac{9}{3})^3}} = \sqrt[3]{1 + \sqrt{-26}} + \sqrt[3]{1 - \sqrt{-26}}.$$

What is one to make of this solution? Since Cardano was suspicious of negative numbers—calling them "fictitious" [11, p. 40]—he certainly had no taste for their square roots—which he named "sophistic negatives" [11, p. 40]. He therefore regarded his formula as inapplicable to equations such as $x^3 = 9x + 2$. Judged by past experience, this was not an unreasonable attitude. For example, to pre-Renaissance mathematicians the quadratic formula could not be applied to $x^2 + 1 = 0$.

All this changed when the Italian Rafael Bombelli came on the scene. In his important book *Algebra* (1572) he applied Cardano's formula to the equation $x^3 = 15x + 4$ and obtained $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. But he could not dismiss this solution, unpalatable as it would have been to Cardano, for he noted—by inspection—that $x = 4$ is also a root of this equation; its other two roots, $-2 \pm \sqrt{3}$, are also real numbers. Here was a paradox: while all three roots of the cubic $x^3 = 15x + 4$ are real, the formula used to obtain them involved square roots of negative numbers—meaningless at the time. How was one to resolve the paradox?

Bombelli had a "wild thought": since the radicands $2 + \sqrt{-121}$ and $2 - \sqrt{-121}$ differ only in sign, the same might be true of their cube roots. He thus let

$$\sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1}, \sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1},$$

and proceeded to solve for a and b by *manipulating these expressions according to the established rules for real variables*. He deduced that $a = 2$ and $b = 1$ and thereby showed that, indeed,

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4.$$

Bombelli had given meaning to the "meaningless". He put it thus [12, p. 19]:

> » It was a wild thought in the judgment of many; and I too for a long time was of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.

Moreover, Bombelli developed a "calculus" for complex numbers, stating such rules as $\left(+\sqrt{-1}\right)\left(+\sqrt{-1}\right) = -1$ and $\left(+\sqrt{-1}\right)\left(-\sqrt{-1}\right) = 1$, and defined addition and multiplication of some of these numbers. These innovations signaled the birth of complex numbers.

But note that this is a *retrospective* view of what Bombelli had done. He did not postulate the existence of a system of numbers—called complex numbers—containing the real numbers and satisfying basic properties of numbers. To him, the expressions he worked with were just that; they were important because they "explained" hitherto inexplicable phenomena. *Square roots of negative numbers could be manipulated in a meaningful way to yield significant results.* This was a bold idea indeed. See [9, 11].

Rafael Bombelli (1526–1572)

The equation $x^3 = 15x + 4$ considered above is an example of an "irreducible cubic", one with rational coefficients, irreducible over the rationals, all of whose three roots are real and distinct. It was shown in the nineteenth century that *any* solution by radicals of such a cubic—not just Cardano's—*must* involve complex numbers [5, 11]. Thus complex numbers are *unavoidable* when determining solutions by radicals of irreducible cubics. It is for this reason that they arose in connection with the solution of *cubic* rather than (as seems much more plausible) *quadratic* equations. Note that the nonexistence of a solution of the quadratic $x^2 + 1 = 0$ was accepted for centuries.

2.6 Growth

Here are several of examples of the penetration of complex numbers into mathematics in the centuries after Bombelli.

As early as 1620, Albert Girard suggested that an equation of degree n *may* have n roots. Such statements of the "Fundamental Theorem of Algebra" were however vague and unclear. For example Descartes, who coined the unfortunate term "imaginary" for the new numbers (Gauss called them "complex"), stated that although one can imagine that every equation has as many roots as is indicated by its degree, no (real) numbers correspond to some of these imagined roots.

Leibniz, who spent considerable time and effort on the question of the meaning of complex numbers and the possibility of deriving reliable results by applying the ordinary laws of algebra to them, thought of complex roots as "an elegant and wonderful resource of divine intellect, an unnatural birth in the realm of thought, almost an amphibium between being and non-being" [12, p. 159].

Complex numbers were used by Johann Heinrich Lambert for map projection, by Jean le Rond d'Alembert in hydrodynamics, and by Euler, d'Alembert, and Lagrange in (incorrect) proofs of the Fundamental Theorem of Algebra.

Euler made important use of complex numbers in, for example, number theory and analysis; he also linked the exponential and trigonometric functions and, arguably, the five most important numbers in mathematics in, respectively, the following two famous formulas: $e^{ix} = \cos x + i \sin x$ and $e^{\pi i} + 1 = 0$. (Euler was the first to designate $\sqrt{-1}$ by "i".) Yet he said of them [10, p. 594]:

> Because all conceivable numbers are either greater than zero, less than zero or equal to zero, then it is clear that the square roots of negative numbers cannot be included among the possible numbers. Consequently we must say that these are impossible numbers. And this circumstance leads us to the concept of such numbers, which by their nature are impossible, and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination.

Even the great Gauss, who in his doctoral thesis of 1797 gave the first essentially correct proof of the Fundamental Theorem of Algebra, claimed as late as 1825 that "the true metaphysics of $\sqrt{-1}$ is elusive" [10, p. 631]. But by 1831 Gauss had overcome these metaphysical scruples and, in connection with a work on number theory, published his scheme for representing them geometrically, as points in the plane. Similar representations by Caspar Wessel in 1797 and by Jean Robert Argand in 1806 had gone largely unnoticed; but when given Gauss' stamp of approval the geometric representation dispelled much of the mystery surrounding complex numbers.

Doubts concerning the meaning and legitimacy of complex numbers persisted for two and a half centuries following Bombelli's work. Yet during that same period these numbers were used extensively. *How can inexplicable things be so useful?* This is a recurrent theme in the history of mathematics. Bombelli's resolution of the paradox dealing with the solution of the cubic $x^3 = 15x + 4$ is an excellent example of this phenomenon.

2.7 Maturity

In the next two decades further developments took place. In 1833 William Rowan Hamilton gave an essentially rigorous algebraic definition of complex numbers as pairs of real numbers, and in 1847 Augustin-Louis Cauchy gave a completely rigorous definition in terms of congruence classes of real polynomials modulo $x^2 + 1$. In this he modelled himself on Gauss' definition of "congruences" for the integers. By the latter part of the nineteenth century most vestiges of mystery and distrust around complex numbers could be said to have disappeared [7].

But this is far from the end of their story. Various developments in mathematics in the nineteenth century gave us deeper insight into the role of complex numbers in mathematics and in other areas. These numbers offer just the right setting for dealing with many problems in mathematics in such diverse areas as algebra, analysis, geometry, and number theory. They have a symmetry and completeness that is often lacking in the real numbers. The following three quotations, by Gauss in 1811, Riemann in 1851, and Jacques Hadamard in the 1890s, respectively, say it well:

> Analysis … would lose immensely in beauty and balance and would be forced to add very hampering restrictions to truths which would hold generally otherwise,
> otherwise, if … imaginary quantities were to be neglected [3, p. 31].
>
> The original purpose and immediate objective in introducing complex numbers into mathematics is to express laws of dependence between variables by simpler operations on the quantities involved. If one applies these laws of dependence in an extended context, by giving the variables to which they relate complex values, there emerges a regularity and harmony which would otherwise have remained concealed [7, p. 64].
>
> The shortest path between two truths in the real domain passes through the complex domain [10, p. 626].

We give brief indications of what is involved in welcoming complex numbers into mathematics.

In algebra, their introduction gave us the celebrated "Fundamental Theorem of Algebra": every equation with complex coefficients has a complex root. The complex numbers offer an example of an "algebraically closed field", relative to which many problems in linear algebra and other areas of abstract algebra have their "natural" formulation and solution.

In analysis, the nineteenth century saw the development of a powerful and beautiful branch of mathematics: "complex function theory". One indication of its efficacy is that a function in the complex domain is infinitely differentiable if once differentiable—which of course is false for functions of a real variable.

In geometry, the complex numbers lend symmetry and generality to the formulation and description of its various branches, including euclidean, inversive, and noneuclidean geometry. For a specific example we mention Gauss' use of complex numbers to show that the regular polygon of seventeen sides is constructible with straightedge and compass.

In number theory, certain Diophantine equations can be solved using complex numbers. For example, the domain consisting of the set of elements of the form $a+b\sqrt{2i}$, with *a* and *b* integers, has *unique factorization,* and in it the Bachet equation $x^2+2=y^3$ factors as $\left(x+\sqrt{2i}\right)\left(x-\sqrt{2i}\right)=y^3$. This greatly facilitates its solution (in integers).

An elementary illustration of Hadamard's dictum that "the shortest path between two truths in the real domain passes through the complex domain" is supplied by the following proof that the product of sums of two squares of integers is again a sum of two squares of integers; that is, given integers a, b, c, and d, there exist integers u and v such that $(a^2+b^2)(c^2+d^2)=u^2+v^2$. For, $\left(a^2+b^2\right)\left(c^2+d^2\right)=\left(a+bi\right)\left(a-bi\right)\left(c+di\right)\left(c-di\right)$ $=\left[\left(a+bi\right)\left(c+di\right)\right]\left[\left(a-bi\right)\left(c-di\right)\right]=\left(u+vi\right)\left(u-vi\right)=u^2+v^2$ for some integers u and v. Try to prove this result without the use of complex numbers and without being given the u and v in terms of a, b, c, and d.

In addition to their fundamental uses in mathematics, complex numbers have become indispensible in science and technology. For example, they are used in quantum mechanics and in electric circuitry. The "impossible" has become not only possible but essential [7].

Problems and Projects

1. Discuss the solution of the quartic by radicals.
2. Research the lives and work of two mathematicians discussed in this chapter.
3. Show how to trisect an angle using trigonometric functions.
4. Discuss the Italian Renaissance, including some of its accomplishments in mathematics (those not discussed in this chapter).
5. Describe the "geometric algebra" of the ancient Greeks.
6. Discuss the algebra of al-Khwarizmi.
7. Show how to solve an elementary problem in euclidean geometry using complex numbers.
8. Discuss the meaning of the logarithms of negative and complex numbers.
9. The "quaternions" (also known as "hypercomplex numbers") contain the complex numbers. Discuss some of their properties that are like those of the complex numbers and some that differ.
10. Show how to resolve the paradox of the irreducible cubic $x^3=15x+4$.

2

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