The Threshold Concept of a Function – A Case Study of a Student's Development of Her Understanding

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This study is a longitudinal case study of a prospective mathematics teacher and her development of understanding of the threshold concept of a function during a semester of studying mathematics. Four interviews within a nine month period are analysed. The results show how the student at the beginning of the semester made linkages to everyday life. After being presented with an abstract definition her understanding changed. At the end of the semester she looked at functions as a platform to stand on and some months later her understanding allowed her to discuss functions as objects. The present study points out the time and work that is needed to transform the understanding of a threshold concept.

Threshold concepts

Studies of learning in higher education have proposed the notion of threshold concepts (Meyer & Land, 2005). In a subject there are several concepts that have a potential to transform the understanding of the subject but also often are problematic for the students to learn. A threshold concept can be seen as a portal to a new and previously unreachable view of the subject area. It is a threshold to pass through, but when passed the threshold the understanding will have been transformed. There are several studies that have argued the case for threshold concepts across a range of subject areas (see e.g. Meyer, Land, & Baillie, 2010). There has also been some discussions about how to conceptualise the transformation when passing the threshold (e.g. Scheja & Pettersson, 2010). However, there are still many research questions regarding the nature of the transformation as students come to understand a threshold concept.

Meyer and Land (2005) characterised threshold concepts as initially troublesome, transformative, integrative and irreversible. However, they also emphasised that, because of individual differences in e.g. prior knowledge, these critical features of a threshold concept will be experienced in varying degrees by students. Threshold concepts tend also to serve as subject boundary markers, and may position students within a liminal space where their understanding is rendered unstable in the oscillation between old and new understanding. It is in the liminal space 'stuck places' may be experienced by students (Meyer & Land, 2006). Understanding a threshold concept will bring about a significant shift in students' perception of a subject or a part thereof. The transformation may be sudden, but it often occurs over a long period. Integrating prior understandings is part of the transformation and understanding the threshold concept will expose previously hidden relations between concepts in the subject area. The change in perspective is unlikely to be forgotten or will be unlearned only by considerable effort.

In the area of calculus e.g. the concepts of function, limit, derivative and integral can be seen as threshold concepts (Pettersson, 2008). Decades of research on students' understandings of these concepts has emphasised the learning problems that students experience (see e.g. Artigue, Batanero, & Kent, 2007). The present case study focuses on one threshold concept, the concept of a function. The study is part of a lager study aiming to explore how university students' understandings of the threshold concept of a function develop during a semester of mathematics studies. The analysis in this case is how one student talks about her understanding and the research question is: How will the student's understanding of a function develop during one semester of mathematics courses?

Research on understanding the concept of function

Much research has been done on students' conceptions of a function. In the early nineties Harel and Dubinsky (1992) edited a book on research about students understandings of function. Previous studies have shown e.g. the differences between the concept definition and students' concept images (Tall & Vinner, 1981) and that a common conception among students is that functions must be represented by an equation and this equation must include a variable (Ferrini-Mundy & Graham, 1994).

Sfard (1991) discussed the duality of mathematical concepts. Several concepts, e.g. function, are introduced as processes. To make it possible to move on and use functions it is also important to understand functions as objects. More recently studies have also shown that students frequently view functions only as processes and the reification to an object is difficult for the learner (Hansson, 2006; Viirman, Attorps, & Tossavainen, 2010). A transformed understanding of the threshold concept of a function will include the ability to view functions both as processes and as objects. The reification will be a part of the transformation.

Method

The present case study is part of a longitudinal study involving 18 first-year prospective mathematics teachers taking introductory courses in calculus at a large Swedish university. In the first semester these students complete general teacher education courses, an introductory course in mathematics and a course in mathematics education. The semester in which data were collected was the students' second and included four mathematics courses: 'Vectors and functions', 'Calculus', 'History of mathematics', and 'Geometry and combinatorics'. Observations were made in the first part of the semester in the course 'Vectors and functions' and at the end of the semester in the 'Calculus' course. Individual interviews with students yielded data in four interviews over a nine months period. Data were also produced from three questionnaires administered over a period of four months. The first two questionnaires asked the students to explain what a function is and to rate their own understanding. The third one also asked if given graphs and formulas represent functions. All of the students that took part in the courses were informed about the research and voluntarily took part of the research. The present author, as the researcher, was not involved in the teaching or in the examination of the courses. The students were also informed that their answers in questionnaires and interviews would not be presented to the lecturers and examiners in a manner that would reveal individual identities.

The case study presented in this paper comprises the analysis of four interviews with one teacher student, Kim (fictitious name). The first three interviews were conducted at the beginning, in the middle and at the end of the semester. The fourth interview was conducted at the end of the next semester. During that semester the student had taken courses in mathematics education, probability and statistics, and general teacher education courses. The interviews were semi-structured and took the student's responses in the questionnaires as a starting point. The duration of the interviews, conducted in a room in the math library, was about 30 minutes each. They were audio recorded and transcribed in full.

A qualitative analysis of interview transcripts was done with a specific focus on how the student's ways of talking about her experiences of understanding changed over time. The transcript was repeatedly read in parallel to listening to the audio file. The analysis applied a context-focused conceptualisation of the development of the student's understanding (Halldén, Scheja, & Haglund, 2007). Emphasis was placed on how Kim developed personal understanding of the learning material by putting it in a particular (cognitive) context or framework where it made sense for her in the perceived circumstances. Notes were made about the contexts the student use and a narrative about the student's development of her understanding was produced. The analysis was discussed with colleagues to improve the validity of the results.

The concept of a function in the textbooks and the teaching

The literature for the course 'Vectors and functions' comprised two compendiums written by mathematicians at the university. In the first chapters in the compendium about functions several issues are presented: complex numbers, geometrical series and the binomial theorem. The next chapters present polynomial and exponential functions. The graphs of these functions are studied by variable substitutions, the derivative is not used. In the last chapter there is a discussion about the concept of a function. Inverse function is defined and exemplified. Logarithmic functions are studied as one example of inverse functions. The definition of a function that is given in this compendium is “A function is a mapping that for all numbers x in a specified set map the number to another number which is called the value of x for the function and is noted $f(x)$” (personal translation). At the end of the course the students in one lecture worked with a text presented as an exercise to read mathematical texts. The text presented the following definition of function: “A function is a subset of the Cartesian product in which all elements x in the domain occur in exactly one pair (x,y)” (personal translation).

The literature in ‘Calculus’ was Persson and Böiers (2010) as well as working sheets including some theory, exercises and reading instructions. The students used the working sheets during the lectures and said that they used the text book just a little. Persson and Böiers provide in Chapter 1 the following definition: “A function is today understood as a rule or a process that in a well-defined and unique way remake (transform) some specified objects to new objects” (p. 7, personal translation). In the working sheets there is no definition or discussion about the concept of a function.

During the teaching there was only in one occasion a discussion about definitions of a function. Related to the reading exercise, the lecturer discussed the definition of a function given in the text used. He used an example where the function was about marks assigned to some of the students in the class, also pointing out that no rule or arithmetic formula was used in that example.

Results

In this section the results from the four interviews will be presented, first each interview separately and then a summary will be given.

Interview 1 – using linkages to everyday life

The teacher student Kim is going to be teacher in mathematics and science in secondary school (grade 7-9). She is a university educated student who had worked as an economist before starting the teacher education programme. She passed the introductory course with good results but disclosed that she had to work a lot to reach the understanding she desired. In the first interview she talked about her impressions of mathematics and her learning during school and previous working life:

> For me mathematics has always been like a drawn curtain and behind that curtain there was a secret I didn't think I could get access to…or that it couldn't make itself available to me [---] and then and as time has passed it has sort of unveiled itself, so now I think I have a pretty, or I don't know, you know…you're confronted with things in life that are relationships, you meet with functions in real life [---] and then you understand and can calculate.

Her talking about mathematics as a secret behind a curtain and its unveiling while confronted with things in real life connected her current understanding mostly to everyday life and less so to the mathematics she had met in upper secondary school.

In the first interview when Kim was asked about what a function was for her she answered:

> Well, I can feel a bit that it is this... you put in something and in there something happens, if you put in a value as in a box and something will happen there according to a recipe, so I think that it is the recipe that is the function for me.

This understanding of a function as a machine is not unusual when asking students about functions. To understand function as a process is an ordinary starting point (cf. Sfard, 1991). However, interesting in Kim's utterance is also her understanding of the function as the recipe or the formula. This will change during the semester, as will be pointed out below.

In the first interview Kim was also asked to give an example of a function:

> For me a function is a relationship between for instance speed and distance, so if I speed up I'll cover a longer distance within a particular time frame. Or a minute tariff on a mobile phone bill, like if I talk longer I use more minutes and so my bill will increase.

In this answer Kim again talked about and provided linkages to everyday life. Even though she talked during the interview about a function as a recipe she did not give any examples or formulas. Her understanding of a function was at this time, in the beginning of the semester, anchored in real life situations.

Interview 2 –accessing abstraction

The second interview was conducted six days after the lecture when the reading exercise presented above was done. In that exercise Kim was presented with an abstract definition of a function and was also given an example by the lecturer. When asked about a definition of a function Kim answered:

> Today I will say it is a relation between two... eh things, still it is [small laugh]... but what's new in my understanding of functions is that there doesn't have to be a rule that defines this relationship; sometimes there's just a

> relationship between two… chosen things and that the function is rather, that it's important that each element of the first given set gets its partner from the second set, and that for me is a new way of understanding functions.

In this utterance Kim displayed two different understandings. She had her understanding from before about function as a relationship, even though her choice of words was slightly different to that used in the first interview. However, she also had a new understanding; an abstract definition of function had in some way been added. She had recognised that it does not need to be a rule defining the relationship. In the previous interview she pointed out that the rule, or in her words the recipe, really is the function. It is also interesting to notice her hesitation when choosing her words "two... eh things". In the examples from everyday life given in the first interview the "things" put in were numbers. Now, after listening to the example given by the lecturer about giving marks (A-F) to students, she may have recognised that the "things" put in must not be numbers.

The second interview indicated, both in direct utterances and in indirect ways from her utterances, that Kim had been influenced by the reading exercise and from the example given by the lecturer. But she also still had her original understanding, and she had not yet completely reconciled this understanding with her new, emerging, understanding. As she put it:

> I think I'm still like in between two understandings of this and I suppose I'm trying to find a way to put them together.

This utterance indicated that Kim, now in the middle of the semester, had entered a liminal space where her understanding of a function was unstable (Meyer & Land, 2006).

Interview 3 – standing on a platform of functions

Before the third interview, conducted at the end of the semester when the calculus course was nearly completed, Kim and the other students had filled in an questionnaire asking if several graphs given in coordinate systems could be a representation for a function or not. Kim had answered correctly on all the graphs and was asked what principles she had used:

> Well, the principle, I look on how many y values I can get for each x value, I kind of look in that direction and then I check, is there only one alternative then it is a function, if there is several alternatives then it could not be a function.

This means that she had no need for checking if there is a rule. It was either no problem for her if the function was discontinuous or if the domain was discrete. She handled functions at this moment, at the end of the semester, in a convenient way. She had passed through the liminal space and presented a new and unified understanding.

Asked what she thought was the most problematic about functions she answered:

> You know all of a sudden I don't think that it's the concept of function that's so difficult any more, but more analysing and being able to juggle algebraically […] but the functions, I think they've suddenly become, well I don't know, a table, a platform, it's what we're doing at the moment and then the analysis of these function suddenly has become the central topic.

Kim's utterance about functions as a platform indicated that she was now moving away from looking at functions as processes. She was able to look up from the concept itself and could use functions as input in other processes.

Interview 4 – establishing an understanding allowing reflection on process-object relations

Interview 4 was conducted at the end of the next semester, aiming at following up the understanding of a function once the students had been able to get a perspective on the courses that discussed functions. Asking Kim again to explain the concept she answered:

> It is still that I all the time remember that I can have a value for *x* and then it has its correspondence, or it kind of reflects on just one value, and that's the way I live with it, and precisely that it's not needed to be continuous, and there don't need to be any patterns that you can follow... I think that was the big case that made me start to look on several other things as functions, it just needs a correspondence and that could be totally arbitrary.

At this time Kim used the understanding following from an abstract definition, for every *x* in a set there is to be exactly one *y*, but she did not use the words from the definition given in the lecture referred to in interview 2. However, her understanding had now become unified; the two understandings she referred to in interview 2 seemed to have got merged into one understanding allowing her to think about functions in a confident way. It could also be noted that Kim in this interview did not use linkages to everyday life. Asked about that, Kim said that she had put the linkages to everyday life away for a while, but also added that she would probably use such linkages again when teaching her students.

In interview 3 Kim talked about functions as a platform. This was followed up in interview 4 when she was asked if she understand functions as objects:

> I: In research, functions are sometimes talked about as objects. Could you feel that functions is kind of, a specific function like the sine function, that it could be like a "thing" in some way, something that you joggle with?
>
> K: No, I don't feel that way at all. It's more like a description of something…well, of course it's an object…I mean it feels as I'm moving along

> something, sort of gliding along a scale in my… in this set I'm allowed to move…and that I can see what happens then, no not [an] object, it's too an abstract…

After first strongly denying the understanding of functions as objects, Kim started to discuss, on her own, her understanding, in talking about gliding on a scale. So she was asked if she instead looks at functions as processes:

> I: Researchers talk also about functions as processes, that it is something you do; you have an input and get an output. If you talk about functions as objects, the process could be inside but you don't need to look at it every time, you understand the function as a thing.
>
> K: In that case I think I'm leaning more towards an object, than a process. It's more like, I can sort of move around within this entity but each value is there all the time, so nothing happens just because I choose a certain x; all those x's are possible to pick all the time, so I think that's the way I see it.

The notions of functions as processes and objects were new for Kim and she had some problems in connecting them to her own understanding. She ended up moving on to an understanding that could arguably be interpreted as an objectification of functions. But as she also pointed out, this was really abstract for her.

Development of Kim's understanding of function

The four interviews revealed four stages in Kim's transformation of her understanding of the threshold concept of a function: using linkages to everyday life, accessing abstraction, standing on a platform of functions, and establishing an understanding allowing reflection on process-object relations. The change observed in her way of talking about a function can be conceptualised in terms of a process of contextualisation in which her repertoire of possible contextualisations (framings) of function was extended. This extended repertoire of contextualisations allowed Kim's initial understanding of function, linked to concrete everyday life examples ("if I speed up I'll cover a longer distance within a particular time frame"), to be gradually enriched to include a more abstract mathematical understanding ("there doesn't have to be a rule that defines this relationship") allowing functions to be seen as objects ("I can sort of move around within this entity").

Discussion

The present case study illustrates and clarifies the complexity of the transformation involved in coming to understand a threshold concept. In Scheja and Pettersson (2010) it was argued that students' shifting of contextualisation is an important part of the development of their conceptualisations of the threshold concepts and that the transformative aspect of threshold concepts could also be conceptualised in this way. This shifting of contextualisation could also be observed in Kim's development of her understanding, from a concrete everyday context to a context allowing functions to be seen as objects. As was argued in Scheja and Pettersson these shifts of contextualisation also allows the student to become gradually more and more aware of the ways of thinking and practising in the subject (McCune & Hounsell, 2005).

The findings also link to the notion of 'liminality' (Meyer & Land, 2006) describing a crucial stage in the process of coming to understand a concept or, indeed, a discipline. Kim seemed to enter this liminal space having two different understandings in parallel in the middle of the semester. She moved on and left this space with a transformed understanding.

Coming to understand functions as objects is not a neat and tidy process; it is highly dynamic and requires hard work (cf. Hansson, 2006; Sfard, 1991; Viirman, et al., 2010). For Kim this process took the whole semester and even several months later she was not comfortable when trying to analyse her understanding in terms of processes and objects. As has been pointed out, the understanding of a threshold concept often takes time to develop. Kim was interviewed over a period of nine months. This kind of longitudinal data is crucial if we want to explore the lengthy process of coming to understand threshold concepts.

It could be argued that Kim is not a typical student. Surely she is not. She has much more experiences of life including studying other subjects and working as an economist. She is also atypical in the way she expressed herself. Mostly it is hard for students to find words to explain their understandings and it is unusual for students to have thought about theirs conceptions. Kim had the words and could express her thoughts about the concept. That made it possible to elicit data about the process of transformation. Presented in this case study is Kim's way of talking about this process, a vignette of a personal journey. And, although it cannot be said that this is the process for every student, there are nevertheless important things to learn from Kim's description of her transformational journey towards the understanding of the threshold concept of a function.

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