A MODEL FOR THE USES OF VARIABLE IN ELEMENTARY ALGEBRA

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This paper presents a theoretical framework we will call the Three Uses of Variable model (3UV model) that can be used as a guideline in the design of tools intended to make diagnostic analysis, to design teaching activities and to analyse textbooks and other teaching materials related to the concept of variable in elementary algebra. Essentially the model consists of a detailed description of the different aspects that underlie a basic understanding of the three main uses of variable in elementary algebra: specific unknown, general number and related variables. We present results that show the possibilities of applying this model in diagnosis, in the analysis of textbooks and in the design of teaching materials.

Antecedents

The development of algebraic knowledge implies the development of the concept of variable. This concept is central as well to the understanding of more advanced topics in mathematics. Its versatility makes it an important object of study. Given the multifaceted character of variable, no matter which starting point is taken, its different uses are very often concurrent within the same problem or situation and this has to be faced during the teaching process. From our point of view, disregarding any of its fundamental aspects in a problem situation may be the cause of many of the difficulties faced by students when working with algebra. In contrast, paying attention to the flexibility of the use of the concept of variable may be the source of a richer understanding.

The Mathematics Education literature, ever since the beginning of the XX century, has highlighted the difficulties students have with the different uses of variable (Thorndike et al.; 1923, Van Engen, 1953; Menger, 1956; Kuchemann, 1980; Wagner, 1983; Matz, 1982; Philipp, 1992; Warren, 1999). Several research studies point out that students encounter tremendous difficulties in grasping the essentials of the notion of variable and in being able to apply flexibly these uses at different levels of abstraction (Matz, 1982; Usiskin, 1988; Trigueros and Ursini, 1999). All these studies have stressed the importance of being able to move flexibly from one use of variable to any of the others, in order to be able to master this concept and work proficiently with algebra.

The understanding of each one of the uses of variable entails the mastering of many different aspects underlying each use at a different levels of abstraction. From our perspective and based on several years of research on the teaching and learning of the concept of variable, an understanding of variables at an elementary level could be described in terms of the following basic capabilities:

- to perform simple calculations and operations with literal symbols;

- to develop a comprehension of why these operations work;
- to foresee the consequences of using variables;
- to distinguish between the different uses of variable;
- to shift between the different uses of variable in a flexible way;
- to integrate the different uses of variable as facets of the same mathematical object.

The 3UV model

Further analysis of students strategies when working with problems related to the different uses of variable and of the mathematical requirements to master this concept led us to the design of a theoretical framework that enable us to analyse students' productions, to design instruments for diagnosis, for research, and to develop didactic approaches. We call this framework the 3UV (three uses of variable) model. It involves a decomposition of variable in its three main uses in elementary algebra: specific unknown, general number and relationship between variables. For each one of these uses, we have stressed different aspects corresponding to different levels of abstraction at which they can be handled. These requirements can be presented in a schematic way as follows:

We consider that the understanding of **variable as unknown** requires to:

U1 - recognise and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem;

U2 - interpret the symbols that appear in equation, as representing specific values;

U3 - substitute to the variable the value or values that make the equation a true statement;

U4 - determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations;

U5 – symbolise the unknown quantities identified in a specific situation and use them to pose equations.

We consider that the understanding of **variable as a general number** implies to be able to:

G1 - recognise patterns, perceive rules and methods in sequences and in families of problems;

G2 - interpret a symbol as representing a general, indeterminate entity that can assume any value;

G3 - deduce general rules and general methods in sequences and families of problems;

G4 - manipulate (simplify, develop) the symbolic variable;

G5 - symbolise general statements, rules or methods;

We consider that the understanding of **variables in functional relationships** (related variables) implies to be able to:

F1 - recognise the correspondence between related variables independently of the representation used (tables, graphs, verbal problems, analytic expressions);

F2 - determine the values of the dependent variable given the value of the independent one;

F3 - determine the values of the independent variable given the value of the dependent one;
F4 - recognise the joint variation of the variables involved in a relation independently of the representation used (tables, graphs, analytic expressions);
F5 - determine the interval of variation of one variable given the interval of variation of the other one;
F6 – symbolise a functional relationship based on the analysis of the data of a problem.
The 3UV model was applied to the analysis of secondary school mathematics textbooks, to the diagnosis of secondary school mathematics teachers conceptions of variable and to the design of school activities involving the different uses of variable at different abstraction levels. In each of these cases an appropriate instrument was designed to serve the specific purpose it was intended for. The instrument was applied and the results were analysed in terms of the 3UV model.

Analysis of textbooks

Method

A grid that takes into account all the aspects involved for the different uses of variable was designed. The grid consists of a table with a column for the number of the lesson, a second column for the theme of the lesson (arithmetic, algebra, geometry, basic statistics, and probability), then each of the uses of variable has its own column subdivided into new columns where the different aspects of the use are taken into account. In Mexico there is an official program for secondary school mathematics education. The books used at this level have to be authorised after a process of evaluation. Three books officially authorised by the ministry of education were analysed. These books correspond to 1st 2nd and 3rd year of secondary education in Mexico. These books consist of 44, 45 and 41 lessons respectively. Each lesson was analysed to see which aspects of the different uses of variable, if any was included. (Benitez, 2001)

Results

The purpose of this analysis was to study the way the concept of variable is included in secondary textbooks, to see which of the uses of variable was stressed and to analyse what kind of relationships were established between the different uses of variable.

The percentages of lessons in which each aspect of the three uses of variable is used in each of the three secondary school textbooks are shown in the following table.

Use	Specific Unknown					General Number					Functional Relationship					
Year	U1	U2	U3	U4	U5	G1	G2	G3	G4	G5	F1	F2	F3	F4	F5	F6
I	77	2	2	77	0	41	27	7	7	4.5	11	7	4.5	7	0	4.5
II	67	15	13	67	11	18	38	4.4	4	13	9	6.7	0	6.7	4.4	6.7
III	73	34	5	73	32	5	56	17	18	19.5	19.5	17.5	10	22	10	19.5

As can be seen in this table, the way in wich variable is used at secondary school is centred mainly in the use of variable as unknown, but its different aspects are not taken into account in a balanced way. The main emphasis is always directed towards U1 and U4. Even if the variable as a general number is considered in some lessons, there is nothing explicit to make sure that the student will develop the ability to differentiate it from the specific unknown, to develop general methods and to work fluently with it. Variable in a functional relationship appears only in few occasions. Even if in several lessons two or three aspects of a specific use of variable or different uses are presented in conjunction, there is no indication helping students to distinguish between them or to shift from one to the other. The 3UV model permits us to see that these textbooks don't provide enough opportunities for the students to integrate the different facets of variable and this lack of flexibility is reflected in the results obtained as they progress through their schooling (Trigueros and Ursini, 1999).

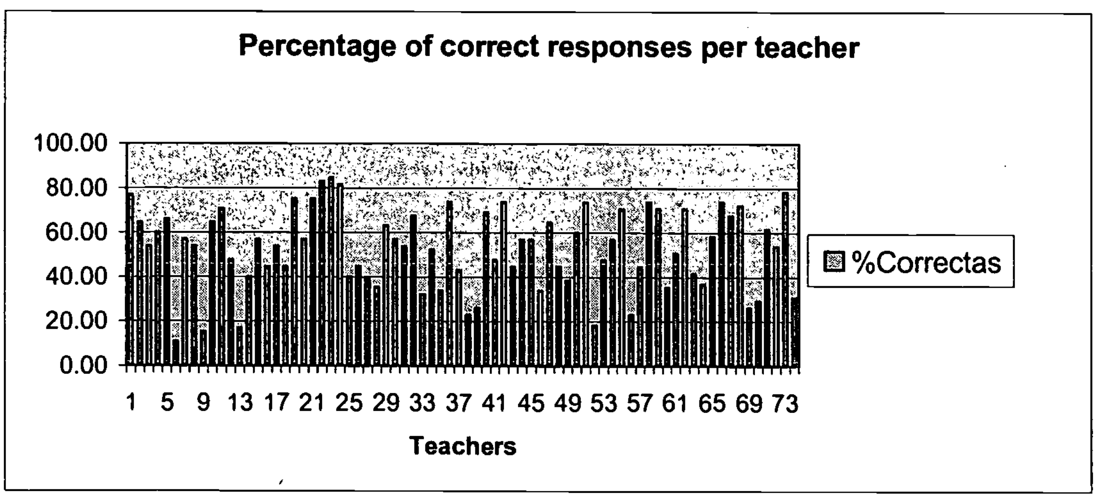
Diagnosis of teachers' conceptions

Method

A questionnaire was used in this case. This questionnaire had been used with students before and the results obtained have been discussed in previous studies (Ursini and Trigueros, 1997). This time the questionnaire was applied to 74 secondary school teachers and their responses were analysed and classified. After reviewing the responses, six teachers were selected to be interviewed. The interviews were also analysed in terms of the model.

Results

The percentage of correct responses per teacher is shown in the following graph



It is striking to see that secondary school teachers results are very similar to those obtained by starting college students (Ursini and Trigueros, 1997) and that their incorrect responses are very similar. Again, none of the questions was answered correctly by all the teachers and no teacher answered correctly all the questions.

High school teachers have problems to differentiate between the uses of variable as unknown and as general number (U2,G2), but in the case of equations they are able to correctly recognise the unknown and find it. Even though their ability to manipulate equations and algebraic expressions (U4,G4) is good, they struggle with the recognition of the unknown and its symbolisation in problem situations (U1, U5). A tendency to avoid the use of algebraic methods and a strong preference for arithmetic approaches was found. Teachers have difficulties to accept open expressions as valid (G2), this is demonstrated by their tendency to complete them by adding an equal sign and thus changing their meaning to that of equations (G2 changed into U2). When an equal sign is present they feel compelled to find a specific result. Their idea of general number can be described more in terms of "any specific number" than in terms of a way to represent a generalisation (G2). They are able to manipulate the general number in different situations (G4) but they always think about it as something that can be determined sooner or later.
The use of variable that teachers have more problems with is variable in a functional relationship. This was also found in the study with college students. Teachers recognise and work with correspondences (F1) and can determine the value of one of the variables when they know the value of the other (F2, F3), however they have trouble with joint variation questions (F4) and their symbolisation (F6). Their description of intervals was always in terms of discrete numbers, that is, they have difficulties with the idea of the continuum in the number line (F5). For them only even and rational numbers are possible as response of several of the problems discussed (Juárez, 2001).

Design of activities

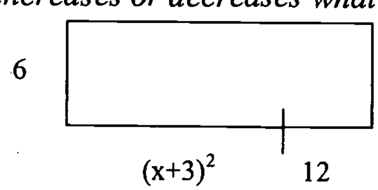
Method

Several activities of different levels of abstraction were developed in terms of the 3UV model. They were designed to take into account the possibility to differentiate and integrate the different uses of variable and to encourage the flexibility in their use. A whole precalculus course for college was designed in terms of the 3UV model.

Results

In order to illustrate the use of the model in the design of activities we show here one of such activities:

For which values of X the area of the following rectangle varies between 168 and 288? If the value of X increases or decreases what happens with the area?



This is not an easy problem. It was proposed for pre-calculus students at college level. The answer to this problem requires the identification of two non contiguous intervals and it implies working with several of the aspects of the three different uses of variable. The variable X has to be recognised as a general number, it has to be manipulated and used in order to obtain new expressions. It has to be recognised that there are unknown quantities involved in the problem that can be determined. Correspondence, variation and determination of intervals are as well involved. Different strategies can be used to solve this problem and students can explore the use of one or another depending of their previous knowledge and mathematical experience.

Some guiding questions that can be used with students in order to help them solve this activity are:

What does X represent in this problem? How many values can it take? The idea of this question is to help the student see the variable X as a general number (G2).

How do you express the area of this rectangle? Students have the tendency to use the memorised formulae for areas without taking into account the specific data given in the problem. This question intends to focus students' attention on the givens and on symbolisation (G5) using the general number and its manipulation (G4) to obtain the expression $6((X+3)^2 + 12)$.

For what values of X the area is equal to 168 and equal to 288? This question can be formulated to help students who use as a starting point for solving an inequality the common strategy that involves solving the equations $6((X+3)^2 + 12) = 168$ and $6((X+3)^2 + 12) = 288$. When students use this strategy they often assume that all the values between the solutions obtained make the inequality true. But this is not always true. With appropriate guide the use of this strategy can help them question its validity. In terms of the uses of variable answering this questions involves posing the quadratic equation (U5), develop the general expressions (G4) and solve both equations (U4). As each equation has two solutions students have to consider that both of them are possible and meaningful in terms of the given problem (U3).

For what values of X the area is bigger than 168 and smaller than 288? For students who used the above mentioned strategy, to answer this question implies the use of the values they already found in order to establish the appropriate intervals ($-9<X<-7$ and $1<X<3$). In other cases, students must form the inequalities (I5) ($168 \leq 6((X+3)^2 + 12) \leq 288$), develop the general expression involved in it (G4) ($168 \leq 6(X^2 + 6X + 9 + 12) \leq 288$) and solve them (I4).

Another possible strategy to solve this question is to relate the expression for the area of the rectangle with a functional relationship between X and area (F1). In this case, answering this question implies defining variation intervals for the dependent variable (F5) and determining, from the graph of the function, the corresponding intervals for the independent variable.

What happens to the area of the rectangle when X varies? The answer of this question involves the recognition of joint variation (F4) of the two variables, determine the values of one of the variables when the values of the other are known (F2). A graphical representation of the relationship can also be used in the solution (F1-F5).

How does the area vary when the value of X increases/decreases? This involves the recognition of the joint variation (F4) and the realisation that while varying X some of the values of the area are a solution to the problem and some of them are not and finding the required intervals for X.

The same problem can be posed using different expressions instead of $(X+3)^2$, that involve different difficulties for the students, for example, for beginning algebra students X or X+2 can be used, and for more advanced students one can introduce more complex expressions such as 1/X.

The detailed description included in the 3UV model is very useful in the design of problems and didactical sequences for algebra courses at different school levels. It can guide the design of didactical sequences to introduce beginning students to the concept of variable so that they can differentiate between its different uses. It can be used to design sequences aiming to foster students' comprehension of variable by introducing relationships between the different aspects of each use and between the different uses, helping them in this way to integrate the different uses of variable in a single concept.

Conclusions

The concept of variable appears in any branch of mathematics. It is widely used also in application of mathematics, but this versatility makes this concept a very difficult one to be mastered by students. The different uses of the concept are on the basis of the difficulties students face when trying to learn algebra.

Research developed during the last thirty years has shown that each use of variable is linked with specific epistemological and didactical obstacles. When algebra is taught taking only one of these aspects as the central focus, the possibility of flexibility and the richness of the relationships between the different uses is lost or obscured and students' understanding of algebra stays limited.

The 3UV model presents a very detailed analysis of each of the main uses of variable in elementary algebra: as a specific unknown, as a general number and in a functional relationship. A lot of research about students' understanding of the concept of variable has been incorporated in the model and it is at the basis of the wide range of possible applications.

The possibility to integrate a collection of related ideas in a single concept, with a specific name, can help students focus their attention on it, manipulate it and use it with ease in applications. Such concepts have rich potential precisely because they carry with them powerful links that enable the users to invoke them to solve problems. If the diverse elements that integrate the concept are not connected sufficiently fluently, it may be very difficult for students to consider such concept as a unit and it may be impossible for them to make links with it. The student can access only some elements of a loosely connected structure, instead of a rich conceptual entity and the possibilities of its invocation are reduced considerably (Skemp, 1970). But identifying the components that have to be linked in that unit can be a very difficult task and cannot be left as something to be done by each teacher. Research is necessary to make this task possible.

The 3UV model is precisely an attempt to fill this necessity in the case of the concept of variable. We have shown here examples of the application of the model to different educational activities: diagnosis, design and analysis of teaching material. These examples show that the analysis of the concept incorporated in the model is detailed enough as to make it possible for researchers and designers to use it in a wide variety of contexts. The possibility to design teaching activities that can help students make the necessary links between the aspects that constitute each one of the uses of variable, and the possibility of identifying them with a specific name may help beginning students to develop a stronger concept of each of the uses of variable. The posterior emphasis in relating the different uses of variable into a single conceptual unity can foster the development of a multifaceted concept of variable.

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