LLMA60 / MMGL61 Analys

Tentamen 11 jan 2013

Lösningar

1.a) $\lim\limits_{x\to\infty} \dfrac{x^2+\sin x}{x^2+\cos x} = \lim\limits_{x\to\infty} \dfrac{1+\frac{\sin x}{x^2}}{1+\frac{\cos x}{x^2}} = \dfrac{1}{1} = 1$

$\left|\dfrac{\sin x}{x^2}\right| \leq \dfrac{1}{x^2} \to 0$, då $x \to 0$

b) $\lim\limits_{x\to-\infty} \dfrac{2x-5}{|13x-11|} = \lim\limits_{x\to-\infty} \dfrac{2x-5}{-13x+11} = \lim\limits_{x\to-\infty} \dfrac{2-\frac{5}{x}}{-13+\frac{11}{x}} = -\dfrac{2}{13}$

Svar: a) 1, b) $-\frac{2}{13}$

2. $f(x) = \dfrac{1}{x^2} - \dfrac{1}{(x-1)^2}$ ej definierad för $x = 0, 1$

$\lim\limits_{x\to 0} f(x) = \infty$, $\lim\limits_{x\to 1} f(x) = -\infty$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$

Grafen skär x-axeln om $\dfrac{1}{x^2} = \dfrac{1}{(x-1)^2} \Leftrightarrow (x-1)^2 = x^2$

$\Leftrightarrow -2x+1=0 \Leftrightarrow x = \frac{1}{2}$.

Vi beräknar $f'(x) = -\dfrac{2}{x^3} + \dfrac{2}{(x-1)^3}$

$f'(x) = 0$: $x^3 = (x-1)^3 \Leftrightarrow -3x^2+3x-1=0$

inga reella lösningar

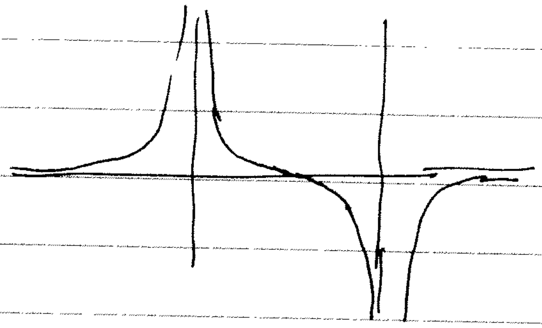
		0		1		x
	+	*	−	*	+	f'
	$\nearrow$	∞	$\searrow$	$-\infty$	$\nearrow$	f

$f''(x) = \dfrac{6}{x^4} - \dfrac{6}{(x-1)^4}$; $f''(x) = 0 \Leftrightarrow (x-1)^4 = x^4 \Leftrightarrow$

$(x^2+(x-1)^2)(2x-1)=0 \iff x=\frac{1}{2}$

Så $x=\frac{1}{2}$ är inflektionspunkt.

Grafen ser ut som



3. a) $$\int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \sin x(1-\cos^2 x)\,dx =$$

$u=\cos x \quad du=-\sin x\,dx$

$x=0 \quad u=1, \quad x=\frac{1}{2}\pi, \; u=0$

$$= -\int_1^0 (1-u^2)\,du = \int_0^1 (1-u^2)\,du =$$

$$= u-\frac{1}{3}u^3 \Big|_0^1 = \frac{2}{3}$$

b) $$\int e^{\sqrt{x}}\,dx = 2\int u e^u\,du \underset{P.I}{=}$$

$u=\sqrt{x} \quad du=\frac{1}{2\sqrt{x}}\,dx$

$x=u^2 \quad dx=2u\,du$

$$2ue^u-2\int e^u\,du = 2(u-1)e^u+C = 2(\sqrt{x}-1)e^{\sqrt{x}}+C$$

Svar: a) $\frac{2}{3}$ b) $2(\sqrt{x}-1)e^{\sqrt{x}}+C$

4. $y'+2xy-x=0$

Integrerande factor e^{x^2}

$$y'e^{x^2}+2xe^{x^2}y=(e^{x^2}y)'=xe^{x^2}$$

$$\Leftrightarrow e^{x^2}y=\int xe^{x^2}\,dx \underset{\substack{u=x^2\\ du=2x\,dx}}{=} \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{x^2}+C$$

$y=Ce^{-x^2}+\frac{1}{2}$

$y(0)=1 \Rightarrow C+\frac{1}{2}=1, \quad C=\frac{1}{2}$ Svar: $y=\frac{1}{2}e^{-x^2}+\frac{1}{2}$.

5. $y = \ln(1-x^2)$ $\quad y' = \frac{-2x}{1-x^2}$

$$\sqrt{1+y'^2} = \sqrt{\frac{(1-x^2)^2}{(1-x^2)^2} + \frac{4x^2}{(1-x^2)^2}} = \sqrt{\frac{1+2x^2+x^4}{(1-x^2)^2}} = \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}$$

Längd: $\int_0^{1/2} \frac{1+x^2}{1-x^2}\,dx = \int_0^{1/2} \frac{x^2-1+2}{1-x^2}\,dx =$

$$\int_0^{\frac{1}{2}} \left(-1 + \frac{2}{1-x^2}\right) dx = \int_0^{\frac{1}{2}} -1 + \frac{1}{1-x} + \frac{1}{1+x}\,dx =$$

$$-x + \ln\frac{1+x}{1-x}\Big|_0^{\frac{1}{2}} = -\frac{1}{2} + \ln\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \ln 3 - \frac{1}{2}$$

Svar: $\ln 3 - \frac{1}{2}$

6. $f(x,y) = (x^2-3y^2)e^{-x^2-y^2}$

vi har $f(x,y) \to 0$ då $x^2+y^2 \to \infty$

$$\nabla f = \langle 2xe^{-x^2-y^2} - 2x(x^2-3y^2)e^{-x^2-y^2}, -6ye^{-x^2-y^2} - 2y(x^2-3y^2)e^{-x^2-y^2} \rangle =$$

$$= \langle 2x(1-x^2+3y^2)e^{-x^2-y^2}, -2y(3+x^2-3y^2)e^{-x^2-y^2} \rangle$$

$$\nabla f = 0 \iff \left.\begin{array}{l} x(1-x^2+3y^2) = 0 \\ y(3+x^2-3y^2) = 0 \end{array}\right\}$$

$x = 0 \Rightarrow y(1-y^2) = 0$; $x \neq 0 \Rightarrow 1-x^2+3y^2 = 0 \Rightarrow 4y = 0$, $1-x^2 = 0$

Så $(x,y) = (0,0)$ $\quad f(0,0) = 0$

$(x,y) = (0,\pm 1)$ $\quad f(0,\pm 1) = -3/e$

$(x,y) = (\pm 1, 0)$ $\quad f(\pm 1, 0) = 1/e$

Svar: största värdet är $1/e$, minsta $-3/e$

7. $\iint_D (x+1)(y-2)\,dA$, $\quad x^2+y^2 \leq 9$

Polära koordinater $\quad x = r\cos\varphi$ $\quad 0 \leq r \leq 3$

$y = r\sin\varphi$ $\quad 0 \leq \varphi \leq 2\pi$

$dA = r\,dr\,d\varphi$

$$\int_0^3\int_0^{2\pi} (r\cos\varphi+1)(r\sin\varphi-2)\,r\,d\varphi\,dr =$$

$$\int_0^3\int_0^{2\pi} \underbrace{r^3\cos\varphi\sin\varphi}_{\frac{1}{2}\sin 2\varphi} + r^2\sin\varphi - 2r^2\cos\varphi - 2r\,d\varphi\,dr =$$

$$= \int_0^3 \underbrace{-\frac{r^3}{4}\cos 2\varphi}_{0} \underbrace{- r^2\cos\varphi}_{0} \underbrace{- 2r^2\sin\varphi}_{0} - 2r\varphi \Big|_0^{2\pi} dr =$$

$$= \int_0^3 -4r\pi\,dr = -2r^2\pi\Big|_0^3 = -18\pi.$$

Svar: -18π.

8. $\frac{d}{dx}\sin 2x = 2\cos 2x$

$\frac{d^2}{dx^2}\sin 2x = -4\sin 2x$

$\frac{d^3}{dx^3}\sin 2x = -8\cos 2x$

$\frac{d^4}{dx^4}\sin 2x = 16\sin 2x = 2^4\sin 2x$

$2013 \equiv 1 \mod 4$

Så $\frac{d^{2013}}{dx^{2013}} = 2^{2013}\cos 2x$

Svar: $2^{2013}\cos 2x$