Gauss (1832) introduced the term "complex number." William Rowan Hamilton (1832) expressed the complex number in the form of a number-couple.

Bombelli continued Cardano's work. From the equation $x^2 + a = 0$, he spoke of "$+\sqrt{-a}$" and "$-\sqrt{-a}$." The special case of this equation, $x^2 + 1 = 0$, affords an excellent approach to i and i^2, as follows: If $x^2 + 1 = 0$, then $x^2 = -1$ and $x = \pm\sqrt{-1}$. Now, if $i = \sqrt{-1}$, then $i^2 + 1 = 0$ when x is replaced by i, and $i^2 = -1$. From this it follows as a good exercise that $i^3 = -\sqrt{-1}$, $i^4 = 1$, $i^5 = \sqrt{-1}, \cdots$, $i^{243} = -\sqrt{-1}$, and so forth.

In his *Algebra* (1673, republished in 1693 in *Opera mathematica*; see /D. E. Smith (c): I, 48/) John Wallis associated "—1600 square perches" with a loss and then supposed this to be in the form of a square with a side [160 square perches = 1 English acre]:

> What shall this side be? We cannot say it is 40, nor that it is -40. (Because either of these multiplied into itself, will make $+1600$; not -1600). But thus rather, that it is $\sqrt{-1600}$, (the Supposed Root of a Negative Square:) or (which is equivalent thereunto) $10\sqrt{-16}$, or $20\sqrt{-4}$, or $40\sqrt{-1}$.

Wallis, Wessel (1798), Jean Robert Argand (1806), Gauss (1813), and others made significant contributions to the understanding of complex numbers through graphical representation, and in 1831 Gauss defined complex numbers as ordered pairs of real numbers for which $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$, and so forth. Wessel's representation is given as follows /D. E. Smith (c): I, 60/:

> Let $+1$ designate the positive rectilinear unit and $+\epsilon$ a certain other unit perpendicular to the positive unit and having the same origin; then the direction angle of $+1$ will be equal to 0°, that of -1 to 180°, that of $+\epsilon$ to 90°, and that of $-\epsilon$ to $-90°$ or 270°. By the rule that the direction angle of the product shall equal the sum of the angles of the factors, we have: $(+1)(+1) = +1$; $(+1)(-1) = -1$; $(-1)(-1) = +1$; $(+1)(+\epsilon) = +\epsilon$; $(+1)(-\epsilon) = -\epsilon$; $(-1)(-\epsilon) = +\epsilon$; $(+\epsilon)(+\epsilon) = -1$; $(+\epsilon)(-\epsilon) = +1$; $(-\epsilon)(-\epsilon) = -1$. From this it is seen that ϵ is equal to $\sqrt{-1}$, and the divergence of the product is determined such that not any of the common rules of operation are contravened.

Of a similar representation it has been said /Bell (d): 234/:

> All this of course proves nothing. *There is nothing to be proved*; we *assign*

> to the symbols and operations of algebra *any meanings whatever* that will lead to consistency. Although the *interpretation* ··· *proves* nothing, it may suggest that there is no occasion for anyone to muddle himself into a state of mystic wonderment over nothing about the grossly misnamed "imaginaries."

A geometric representation credited to Wessel and Argand independently is based on the geometric principle that the altitude to the hypotenuse of a right triangle is a mean proportional between the segments into which the altitude divides the hypotenuse. In Figure [76]-1, $OD_1 = d_1 = +1$, $OD_2 = d_2 = -1$. $\angle D_1RD_2$ is a right angle, and $OR = d$. Then $d_1 : d = d : d_2$. Now $d = \sqrt{d_1 d_2} = \sqrt{+1 \cdot -1} = \sqrt{-1} = i$.

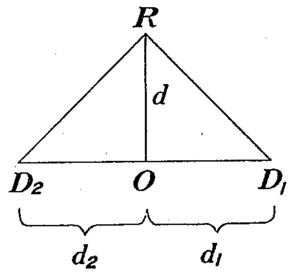

Figure [76]-1

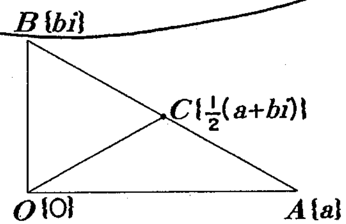

Figure [76]-2

Some interesting geometric proofs can result from the representation of the complex number $a + bi$ by the point in the plane with rectangular coordinates a and b. An example is the proof that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices. In Figure [76]-2 O is the vertex of the right angle of right triangle AOB, and C is the midpoint of the hypotenuse AB. Using the coordinates in the figure,

$$OC = |\tfrac{1}{2}(a + bi) - 0| = \tfrac{1}{2}|(a + bi)| = \tfrac{1}{2}\sqrt{a^2 + b^2},$$

and