Lösningar 20101112

1. (a) $V(X) = E[X^2] - E[X]^2 = \dots$

$E[X^2] = 0^2 \cdot (1-c) + 1^2 \cdot c = c$

$E[X] = 0 \cdot (1-c) + 1 \cdot c = c$

$\Rightarrow V(X) = c - c^2 = \boxed{c(1-c)}$

(b) $E[2^X] = 2^0 \cdot (1-c) + 2^1 \cdot c = 1 - c + 2c = \boxed{1+c}$

(c) $E[X^3] = 0^3 \cdot (1-c) + 1^3 \cdot c = \boxed{c}$

2. (a) $P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\geq 0} \leq P(A) + P(B)$

(b) Använd att $B = A \cup (B \setminus A)$ där $A \cap (B \setminus A) = \emptyset$.

$\Rightarrow P(B) = P(A) + \underbrace{P(B \setminus A)}_{\geq 0} \Rightarrow P(A) \leq P(B)$

(c) $P(A^c \cap B^c) = P(B^c) - P(A \cap B^c) = P(B^c) - P(B^c \cap A)$

$= P(B^c) - (P(A) - P(B \cap A)) \underset{\text{oberoende!}}{=} P(B^c) - P(A)(1 - P(B))$

$= 1 - P(B) - P(A)(1 - P(B)) = (1 - P(A))(1 - P(B)) =$

$= P(A^c) P(B^c)$

3. (a) $\hat{\lambda} = \frac{1+4+0+1+0+0+0+0+1+1}{10} = \frac{8}{10} = \boxed{0.8}$

(b) $E[\hat{\lambda}] = E\left[\frac{X_1 + \dots + X_{10}}{10}\right] = \frac{1}{10} E[X_1 + \dots + X_{10}] =$

$= \frac{1}{10}(\underbrace{E[X_1]}_{=\lambda} + \dots + \underbrace{E[X_{10}]}_{=\lambda}) = \frac{1}{10} \cdot 10 \cdot \lambda = \boxed{\lambda}$

(c) $V(\hat{\lambda}) = \frac{1}{100} V(X_1 + \dots + X_{10}) = \frac{1}{100} (\underbrace{V(X_1)}_{=\lambda} + \dots + \underbrace{V(X_{10})}_{=\lambda}) =$

$= \frac{1}{100} \cdot 10 \cdot \lambda = \boxed{\frac{\lambda}{10}}$

(d) Tabell s. 387 för $\lambda = 0.8$, $x = 0 \Rightarrow P(X = 0) = \boxed{0.4493}$

4. $P(\text{sjuk}) = \frac{1}{10000} = 0.0001$

$P(\text{pos}|\text{sjuk}) = 0.99$, $P(\text{pos}|\text{frisk}) = 0.01$

(Bayes sats: $P(\text{sjuk}|\text{pos}) = \frac{P(\text{pos}|\text{sjuk})P(\text{sjuk})}{P(\text{pos}|\text{sjuk})P(\text{sjuk}) + P(\text{pos}|\text{frisk})P(\text{frisk})}$

$= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.01 \cdot 0.9999} = \boxed{0.0098 \approx 1\% \text{ risk}}$

5. Antal ja-svar antas $Bin(100, p)$-fördelade, där p är denna sanna proportionen ja-svar i hela populationen.

Skatta p med $\hat{p} = 34/100 = 0.34$

Eftersom $n\hat{p}(1-\hat{p}) \geq 10$ så använder vi normal-approximation och får approx konfidensintervall

$$\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad n = 100, \quad z_{\alpha/2} = z_{0.025} = 1.96$$

$\Rightarrow I_p = [0.2472, 0.4328]$

6. Anta $X \sim N(\mu, \sigma)$.

$x_1 = 240.6$, $x_2 = 238.6$, $x_3 = 241.2$, $x_4 = 240.0$, $x_5 = 240.2$

$\hat{\mu} = \bar{x} = 240.08$

$\hat{\sigma}^2 = S^2 = 1.0920 \Rightarrow S = 1.0450$

$100 \cdot (1-\alpha)$ % konf int fås som

$$I_\mu = \bar{x} \pm t_{(n-1)\alpha/2} \frac{S}{\sqrt{n}} \qquad n = 5, \ \alpha = 0.01$$

$t(4)_{0.005} = 4.604$ enl tabell

$$\Rightarrow I_\mu = 240.08 \pm 4.604 \cdot \frac{1.0450}{\sqrt{5}} = [237.9, 242.2]$$

$240 \in I_\mu$ så obs. talar ej emot, nollhyp behålls på 1%-nivån.

7 (a) $E[XY] \approx \frac{1}{n} \sum_{i=1}^{10} x_i y_i = 1671.3$; $\bar{x} = 41.50$, $\bar{y} = 37.56$

$S_x = 22.11$, $S_y = 6.24$

$$\Rightarrow \hat{\rho}(X, Y) = \frac{1671.3 - 41.50 \cdot 37.56}{22.11 \cdot 6.24} = \boxed{0.8159}$$

(b) $Y = aX$; $C(X, Y) = C(X, aX) = E[aX^2] - E[X]E[aX] =$
$= a(E[X^2] - E[X]^2) = aV(X)$.

$\sigma_X^2 = V(X)$; $\sigma_Y^2 = V(aX) = a^2 V(X)$

$$\Rightarrow \rho(X, Y) = \frac{aV(X)}{\sqrt{V(X) \cdot a^2 V(X)}} = \frac{aV(X)}{aV(X)} = \underline{1}$$