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1. a. $V(X) = E[X^2] - E[X]^2 = \left(0^2 \cdot (1-p) + 1^2 \cdot p\right) - \left(0 \cdot (1-p) + 1 \cdot p\right)^2 =$
$= p - p^2 = \underline{p(1-p)}.$

b. $E[2^X] = 2^0 (1-p) + 2^1 \cdot p = 1 - p + 2p = 1 + p$

c. $E[X^3] = 0^3 \cdot (1-p) + 1^3 \cdot p = p$

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = P(A) + P(B) - \underbrace{P(A \cup B)}_{\leq 1} \geq P(A) + P(B) - 1$

ty sannolikhet

3. Lagen om total sannolikhet:

$P(A) = P(A|C)\,P(C) + P(A|C^c)\,P(C^c)$

$P(B) = P(B|C)\,P(C) + P(B|C^c)\,P(C^c)$

Så

$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c) > P(B|C)P(C) + P(B|C^c)P(C^c) = P(B)$

$\Rightarrow P(A) > P(B)$

4. Alt 1: $P(A^c \cap B^c) = 1 - P(A \cup B) =$

$= 1 - \left(P(A) + P(B) - P(A \cap B)\right) = 1 - P(A) - P(B) + P(A)P(B) =$

$= 1 - (1 - P(A^c)) - (1 - P(B^c)) + (1 - P(A^c))(1 - P(B^c)) =$

$= \cancel{1} - \cancel{1} + \cancel{P(A^c)} - \cancel{1} + \cancel{P(B^c)} + \cancel{1} - \cancel{P(A^c)} - \cancel{P(B^c)} + P(A^c)P(B^c) = P(A^c)P(B^c)$

Alt 2: $P(A^c \cap B^c) = P(B^c) - P(A \cap B^c) = P(B^c) - P(B^c \cap A) =$

$= P(B^c) - \left(P(A) - P(B \cap A)\right) = P(B^c) - P(A)\left(1 - P(B)\right) =$

$= 1 - P(B) - P(A)\left(1 - P(B)\right) = \left(1 - P(A)\right)\left(1 - P(B)\right) = P(A^c)\,P(B^c)$

5. $C(X+Y, X-Y) = E[(X+Y)(X-Y)] - E[X+Y]E[X-Y] =$
$= E[X^2 + \cancel{XY} - \cancel{XY} - Y^2] - (E[X]^2 + E[\cancel{XY}] - E[\cancel{XY}] - E[Y]^2) =$
$= E[X^2] - E[X]^2 - (E[Y^2] - E[Y]^2) =$
$= V(X) - V(Y) = V(X) - V(X) = \underline{0}$.

6. Antal ja-svar antas $Bin(100, p)$-fördelade, där p är den sanna proportionen ja-svar i hela populationen.

Skatta p med $\hat{p} = 19/100 = 0.19$

Eftersom $n\hat{p}(1-\hat{p}) \approx 15 \geq 10$ använder vi normalapproximationen och får approximativt konfidensintervall för p:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad n=100 \quad z_{\alpha/2} = z_{0.025} = 1.96$$

$\Rightarrow I_p = [0.1131, 0.2669]$

dvs konfidensintervallet innehåller värden > 0.25, så det ger ingen grund för påståendet $p < 0.25$.

7. Skatta medelvärden och varianser

$\bar{x}_1 = 58.492 \quad \bar{x}_2 = 59.96$

$s_1^2 = 0.9119 \quad s_2^2 = 0.8819$

'pooled variance': $\hat{\sigma}^2 = s_p^2 = \dfrac{(5-1)s_1^2 + (4-1)s_2^2}{5+4-2} = 0.8990$

$100(1-\alpha)\%$ konfidensintervall:

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2}\sqrt{s_p^2\left(\frac{1}{5}+\frac{1}{4}\right)} \qquad \alpha = 0.05 \ (95\%)$$

$t_{0.025}$ tas ur t-fördelning med $5+4-2=7$ frihetsgrader

Tabell $\Rightarrow$ $t_{0.025} = 2.365$

Konf. int $I_{\mu_1-\mu_2} = [-2.9723, 0.0363]$

8. $X \sim Bin(n=100,\ p=2/3)$

$np(1-p) = 100 \cdot \frac{2}{3} \cdot \frac{1}{3} \geq 10$; Så normalapprox:

$$X \overset{approx}{\sim} N\left(np, \sqrt{np(1-p)}\right) = N\left(\frac{200}{3}, \frac{10\sqrt{2}}{3}\right)$$

$$P(X \geq 70) = P\Bigg(\underbrace{\frac{X - 200/3}{10\sqrt{2}/3}}_{W \sim N(0,1)} \geq \frac{70 - \frac{200}{3}}{10\sqrt{2}/3}\Bigg)$$

$$= P\left(W \geq \frac{1}{\sqrt{2}}\right) = 1 - P\left(W \leq \frac{1}{\sqrt{2}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{2}}\right)$$

$$= 1 - \underbrace{\Phi(0.707)}_{\text{tabell} \Rightarrow 0.7611} = 0.2389$$

9a. $$E[\bar{X}] = E\left[\frac{X_1 + \dots + X_n}{n}\right] = \frac{1}{n} E[X_1 + \dots + X_n] =$$

$$= \frac{1}{n}\Big(\underbrace{E[X_1]}_{\mu} + \dots + \underbrace{E[X_n]}_{\mu}\Big) = \frac{1}{n} n\mu = \underline{\mu}.$$

b. Betrakta en term $(X_i - \bar{X})^2 = X_i^2 - 2X_i\bar{X} + \bar{X}^2 =$

$$= X_i^2 - 2X_i \frac{1}{n}(X_1 + \dots + X_n) + \frac{1}{n^2}(X_1 + \dots + X_n)^2 =$$

$$= X_i^2 - \frac{2}{n}X_i^2 - \frac{2}{n}\sum_{j \neq i} X_i X_j + \frac{1}{n^2}\left(\sum_j X_j^2 + \sum_j \sum_{k \neq j} X_j X_k\right)$$

$$E[(X_i - \bar{X})^2] = E[X_i^2] - \frac{2}{n}E[X_i^2] - \frac{2}{n}\sum_{j \neq i} E[X_i]E[X_j] + \dots$$

9b. forts

$$\ldots + \frac{1}{n^2}\left(\sum_j E[X_j^2] + \sum_j\sum_{k\neq j} E[X_j]E[X_k]\right) =$$

$$= E[X_i^2] - \frac{2}{n}E[X_i^2] - \frac{2}{n}(n-1)E[X_i]^2 + \ldots$$

$$+ \frac{1}{n^2}\left(nE[X_j^2] + n(n-1)E[X_i]^2\right) =$$

$$= \left(1 - \frac{2}{n} + \frac{1}{n}\right)E[X_i^2] + \left(-\frac{2(n-1)}{n} + \frac{n-1}{n}\right)E[X_i]^2$$

$$= \frac{n-1}{n}\left(E[X_i^2] - E[X_i]^2\right) = \frac{n-1}{n}V(X_i) = \underline{\frac{n-1}{n}\sigma^2}$$

$$\Rightarrow E[S^2] = \frac{1}{n-1}\sum_{i=1}^{n} E[(X_i - \bar{X})^2] =$$

$$= \frac{1}{\cancel{n-1}} \cdot \cancel{n} \cdot \frac{\cancel{n-1}}{\cancel{n}}\sigma^2 = \underline{\underline{\sigma^2}}$$, alltså väntevärdesriktig