8.4 The Impossible Star of David

How many ways can an n-pointed star be drawn from n straight lines without lifting the pen? An n-pointed star has of course n outer corners. For simplicity we assume that these corners lie equidistantly on a circle. Connecting corners that are adjacent on the circle produces a polygon, not a star. To obtain a star we must connect each corner with one of its $n-2$ non-neighbors. For $n = 6$, we get the six-pointed Star of David: two superimposed triangles.

But suppose we are asked to draw a six-pointed star in six straight consecutive strokes. The Star of David then becomes impossible (the reader is invited to try the following himself). Suppose the points are numbered 1,2,3,4,5,6 and we connect 1 with 3 with 5 with 1. But that is not a six-pointed star; that is a triangle and points 2,4,6 have been left out. Suppose we skip *two* points. Then we connect 1 with 4 with 1, and we have missed four points. Suppose we permit irregular skippings and connect, for example, 1 with 3 with 6 with 2 with 4 — but now we cannot skip anymore because the only missing point is the adjacent point 5.

No matter how hard we try, a six-pointed star cannot be completed in this way. Yet it is easy to draw a five-pointed star by five straight lines without lifting the pen: connect point 1 with 3 with 5 with 2 with 4 with 1. And there are even *two* seven-pointed stars: skipping *one* point at a time, connect point 1 with 3 with 5 with 7 with 2 with 4 with 6 with 1 (Fig. 8.1); or, skipping *two* points at a time, connect point 1 with 4 with 7 with 3 with 6 with 2 with 5 with 1 (Fig. 8.2).

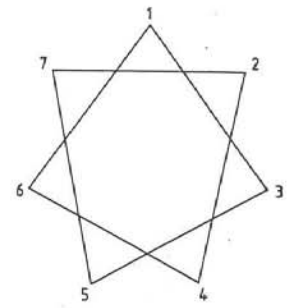

Fig. 8.1 One of two possible regular seven-pointed stars. It is based on the fact that 7 is coprime with 2 (and 5)

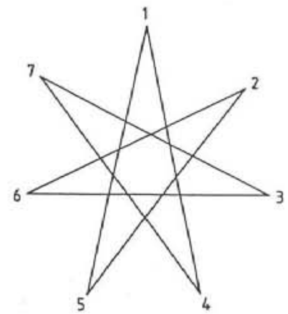

Fig. 8.2. The other possible seven-pointed stars based on the coprimality of 7 with 3 (and 4)

When is a star possible? If $k-1$ is the number of points skipped, then k must be greater than 1 and smaller than $n-1$ (or we would get a polygon) and k must be coprime with n, the number of corners. Because $\phi(5) = 4$, there are four values of k such that $(k,5) = 1$, of which only two differ from 1 or $n-1$: $k = 2$ and $k = 3$.

The two stars drawn with $k = 2$ and $k = 3$ are indistinguishable once the drawing is completed. Thus, there is only one possible five-pointed star.

From this discussion it should be clear that for general n, the following holds:

Number of possible stars = [redacted] (8.23)

Let us try this formula for $n = 6$. With $\phi(6) = 2$, we get *no* stars — as we have already discovered by trial and error.

For $n = 7$ we get, with $\phi(7) = 6$, [redacted] = 2 stars. Check! (See Figs. 8.1, 2.)

Question: for which n do we get 6 different stars? Answer: for *no* n — the number [redacted] = 14 is a so-called *nontotient:* $\phi(n)$ can never equal 14. (Can the reader show this?)

The number 6 seems to have an unlucky star: not only is a singly connected six-pointed star impossible, but there is no n for which exactly six different stars are possible. Other nontotients are 26 and 34. What is the general condition for a number to be a nontotient?