MAL600 Flervariabelanalys 16 aug 04

1. Kedjeregeln på $h: t \longmapsto (t^2, t^3) \longmapsto f(t^2, t^3)$ ger

$$h'(t) = 2t \cdot f'_x(t^2,t^3) + 3t^2 \cdot f'_y(t^2,t^3)$$ och

$$h''(t) = 2 \cdot f'_x(t^2,t^3) + 2t \cdot \left(2t \cdot f''_{xx}(t^2,t^3) + 3t^2 \cdot f''_{xy}(t^2,t^3)\right) +$$
$$+ 6t \cdot f'_y(t^2,t^3) + 3t^2 \cdot \left(2t \cdot f''_{yx}(t^2,t^3) + 3t^2 \cdot f''_{yy}(t^2,t^3)\right)$$

och eftersom $t=-1$ ger $(t^2, t^3) = (1,-1)$ får vi

$$h''(-1) = 2 \cdot 1 + 2 \cdot (-1)\left(2 \cdot (-1) \cdot 1 + 3 \cdot 1 \cdot 1\right) + 6 \cdot (-1) \cdot 1 + 3\left(2 \cdot (-1) \cdot 1 + 3 \cdot 1 \cdot 1\right) = -3.$$

2. $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \quad (+n\pi) \end{cases} \quad (x \neq 0)$$

$$z'_x = z'_r r'_x + z'_\theta \theta'_x = z'_r \cdot \frac{x}{\sqrt{x^2+y^2}} + z'_\theta \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2}$$

$$z'_y = z'_r r'_y + z'_\theta \theta'_y = z'_r \cdot \frac{y}{\sqrt{x^2+y^2}} + z'_\theta \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$x z'_x + y z'_y = z'_r \cdot \frac{x^2+y^2}{\sqrt{x^2+y^2}} + z'_\theta \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x} + \frac{y}{x}\right) = z'_r \cdot \sqrt{x^2+y^2} = r z'_r.$$

Alternativt kan man beräkna

$$\begin{pmatrix} r'_x & r'_y \\ \theta'_x & \theta'_y \end{pmatrix} = \begin{pmatrix} x'_r & x'_\theta \\ y'_r & y'_\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

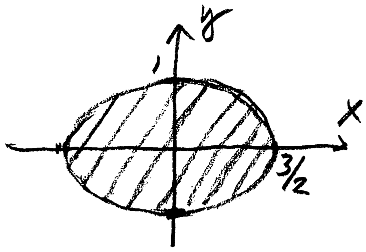
3. $$\begin{cases} f'_x = 3x^2y + y^3 - y = y(3x^2+y^2-1) = 0 \iff \text{I. } y=0 \text{ el. II. } 3x^2+y^2=1 \\ f'_y = x^3 + 3xy^2 - x = x(x^2+3y^2-1) = 0 \iff \text{III. } x=0 \text{ el. IV. } x^2+3y^2=1 \end{cases}$$

I & III: $(0,0)$ I & IV: $(\pm 1, 0)$ II & III: $(0, \pm 1)$

II & IV $\Rightarrow 2x^2 - 2y^2 = 0 \Rightarrow x^2 = y^2 = \frac{1}{4}$; $\pm\left(\frac{1}{2}, \pm\frac{1}{2}\right)$.

	(x,y)	$(0,0)$	$(\pm 1,0), (0,\pm 1)$	$\pm(\frac{1}{2},\frac{1}{2})$	$\pm(\frac{1}{2},-\frac{1}{2})$
$A = f''_{xx}$	$6xy$	0	0	$3/2$	$-3/2$
$B = f''_{xy}$	$3x^2+3y^2-1$	-1	2	$1/2$	$1/2$
$C = f''_{yy}$	$6xy$	0	0	$3/2$	$-3/2$
$AC-B^2$		-1	-4	2	2
		sadel	sadel	lok. min. $-1/8$	lok. max. $1/8$

4.



$$\begin{cases} f'_x = 24x^2 - 24x = 24x(x-1) \\ f'_y = 162y - 27y^2 = 27y(6-y) \end{cases}$$

Stationära punkter i det inre är $(0,0)$ och $(1,0)$.

*) Randen: $g(x,y) = 4x^2 + 9y^2 = 9$; $\nabla g = (8x, 18y)$ så

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 24x(x-1) = 8\lambda x \\ 27y(6-y) = 18\lambda y \end{cases}$$

I. $x = 0 \Rightarrow 9y^2 = 9 \Rightarrow y = \pm 1$

II. $y = 0 \Rightarrow 4y^2 = 9 \Rightarrow x = \pm 3/2$

III. $6(x-1) = 2\lambda = 3(6-y) \Rightarrow 2x - 2 = 6 - y \Leftrightarrow 2x + y = 8$,

men $x \leq 3/2$, $y \leq 1$, så $2x + y \leq 3 + 1 = 4 < 8$; III gäller ej.

Möjliga extremvärden är alltså

$f(0,0) = 0$, $f(1,0) = -4$, $f(3/2, 0) = 0$, $f(-3/2, 0) = -54 = \min$

$f(0,1) = 72$, $f(0,-1) = 90 = \max$.

5. I polära koordinater ges I:a kvadranten av $0 \leq \theta \leq \frac{\pi}{2}$,

så $\iint_D x^2 e^{-x^2-y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}} \left(\int_0^\infty r^2 \cos^2\theta\, e^{-r^2} r\,dr \right) d\theta =$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \cdot \int_0^\infty \underbrace{r^2}_{\text{der.}} \cdot \underbrace{re^{-r^2}}_{\text{int.}}\,dr = \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta \cdot \left(\left[-\tfrac{r^2}{2} e^{-r^2} \right]_0^\infty + \int_0^\infty r e^{-r^2}\,dr \right) =$$

$$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{2}} \cdot \left(0 + \left[-\tfrac{1}{2} e^{-r^2} \right]_0^\infty \right) = \frac{\pi}{4} \cdot \frac{1}{2} = \frac{\pi}{8}.$$

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*) Man kan också parametrisera randen: $\begin{cases} x = \frac{3}{2}\cos t \\ y = \sin t \end{cases}$

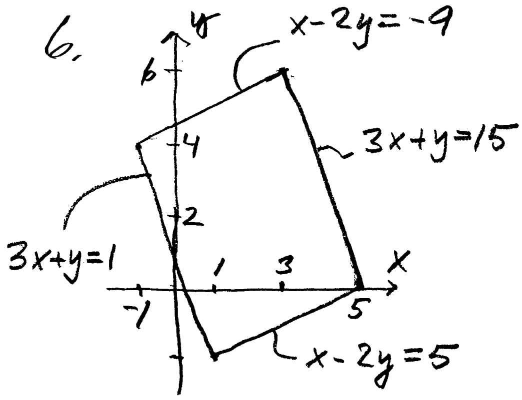
$$g(t) = f\left(\tfrac{3}{2}\cos t, \sin t\right) = 27\cos^3 t - 27\cos^2 t + 81\sin^2 t - 9\sin^3 t$$

$$g'(t) = -81\cos^2 t \sin t + 54\cos t \sin t + 162 \sin t \cos t - 27\sin^2 t \cos t =$$

$$= 27 \sin t \cos t \underbrace{(8 - 3\cos t - \sin t)}_{> 8-3-1 = 4,\ \text{så} \neq 0}$$

$g'(t) = 0 \Leftrightarrow \sin t = 0$ el. $\cos t = 0$, dvs. $y = 0$ el. $x = 0$.

6.



Låt $\begin{cases} u = x-2y \\ v = 3x+y \end{cases}$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7 \text{ så } \frac{d(x,y)}{d(u,v)} = \frac{1}{7}.$$

$D \Leftrightarrow D'$: $-9 \le u \le 5$, $1 \le v \le 15$

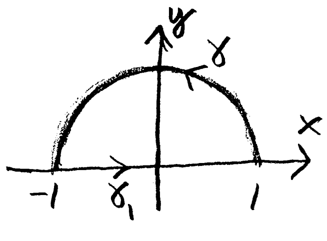
$$\iint_D 7x\cos(x-2y)\,dx\,dy = \iint_{D'} (u+2v)\cos u \cdot \frac{1}{7}\,du\,dv =$$

$$= \frac{1}{7}\int_{-9}^{5} \cos u \left(\int_1^{15} (u+2v)\,dv\right) du = \frac{1}{7}\int_{-9}^{5} \cos u \,\Big[uv+v^2\Big]_{v=1}^{v=15} du =$$

$$= \frac{1}{\cancel{7}}\int_{-9}^{5} \underbrace{(\overset{2}{\cancel{14}}u + \overset{32}{\cancel{224}})}_{\text{der}} \underbrace{\cos u}_{\text{int}}\,du = \Big[(2u+32)\sin u\Big]_{-9}^{5} - 2\int_{-9}^{5} \sin u\,du =$$

$$= 42\sin 5 - 14\sin(-9) + \Big[2\cos u\Big]_{-9}^{5} = 42\sin 5 + 14\sin 9 + 2\cos 5 - 2\cos 9.$$

7.



Låt γ_1 vara sträckan från $(-1,0)$ till $(1,0)$ och D den av $\gamma+\gamma_1$ inneslutna halvcirkelskivan.

På γ_1 är $y=0$ och $dy=0$ så $\int_{\gamma_1} P\,dx + Q\,dy = 0$ och

$$\int_\gamma (x^2y + \tfrac{1}{3}y^3 + ye^{xy})\,dx + (x + xe^{xy})\,dy = \int_{\gamma+\gamma_1} =$$

$$= \iint_D \left(1 + e^{xy} + xye^{xy} - (x^2+y^2+e^{xy}+xye^{xy})\right) dx\,dy =$$

$$= \iint_D \left(1-(x^2+y^2)\right) dx\,dy = \int_0^\pi \left(\int_0^1 (1-r^2)r\,dr\right) d\theta =$$

$$= \int_0^\pi d\theta \cdot \int_0^1 (r-r^3)\,dr = \pi \cdot \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{4}.$$

8. $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$

$f(x,0) = x \Rightarrow f'_x(x,0) = 1$ för alla x

$f(0,y) = -y \Rightarrow f'_y(0,y) = -1$ för alla y

så $\operatorname{grad} f(0,0) = (1,-1)$.

Om $\vec{v} = (\alpha, \beta)$, $\alpha^2+\beta^2 = 1$, så

$$f'_{\vec{v}}(0,0) = \lim_{t\to 0} \frac{f(\alpha t, \beta t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{\alpha^3 t^3 - \beta^3 t^3}{\alpha^2 t^2 + \beta^2 t^2}}{t} =$$

$$= \frac{\alpha^3-\beta^3}{\alpha^2+\beta^2} = \alpha^3 - \beta^3.$$

$\vec{v} \cdot \operatorname{grad} f(0,0) = (\alpha,\beta)\cdot(1,-1) = \alpha - \beta$ är lika med $f'_{\vec{v}}(0,0)$ om och endast om

$\alpha - \beta = \alpha^3 - \beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2) = (\alpha-\beta)(1+\alpha\beta)$

$\Leftrightarrow (\alpha-\beta)\alpha\beta = 0 \Leftrightarrow \alpha = \beta$, $\alpha = 0$ el. $\beta = 0$,

dvs. riktningsderivatan tas i riktningen $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ eller är en av de partiella derivatorna.

f är tydligen inte differentierbar i $(0,0)$, ty då skulle $f'_{\vec{v}}(0,0) = \vec{v} \cdot \operatorname{grad} f(0,0)$ gälla för alla enhetsvektorer $\vec{v}$.