MAL600 Flervariabelanalys 10 jan 2005

1. $f(x,y,z) = x^2 + xyz + yz^3$, $g(x,y,z) = x+y+z-3x^2z^3 - yz^2$.
(1,1,1) tillhör skärningen mellan $f(x,y,z)=3$ och $g(x,y,z)=-1$.
$\nabla f(1,1,1) = (2x+yz, xz+z^3, xy+3yz^2)|_{(1,1,1)} = (3,2,4)$.
$\nabla g(1,1,1) = (1-6xz^3, 1-z^2, 1-9x^2z^2-2yz)|_{(1,1,1)} = (-5,0,-10) = -5(1,0,2)$.
$\begin{vmatrix} e_x & e_y & e_z \\ 3 & 2 & 4 \\ 1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ så tangent $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$.

2. $\begin{cases} u = x \\ v = y^2+2xy \end{cases}$ $\quad z'_x = z'_u u'_x + z'_v v'_x = z'_u + 2y z'_v$
$\quad z'_y = z'_u u'_y + z'_v v'_y = (2y+2x) z'_v$

a) $0 = (x+y)z'_x - yz'_y = (x+y)z'_u + 2y(x+y)z'_v - y(2y+2x)z'_v = (x+y)z'_u$
$x,y \geq 0 \Rightarrow z'_u = 0 \Rightarrow z(u,v) = g(v) \Rightarrow z(x,y) = g(y^2+2xy)$.
b) $y^4 = z(0,y) = g(y^2) \Rightarrow g(t) = t^2 \ (t \geq 0) \Rightarrow z = (y^2+2xy)^2$.

3. $f(x,y) = x^2 e^{-x^2-y^2}$
$\begin{cases} f'_x = 2xe^{-x^2-y^2} - 2x^3e^{-x^2-y^2} = 2x(1-x^2)e^{-x^2-y^2} \\ f'_y = -2x^2y\, e^{-x^2-y^2} \end{cases}$
$f'_x = f'_y = 0 \Leftrightarrow x = 0$ eller $(x,y) = (\pm 1, 0)$

	(x,y)	$(0,y)$	$(\pm 1, 0)$
$A = f''_{xx}$	$(2-6x^2-4x^2+4x^4)e^{-x^2-y^2}$	$2e^{-y^2}$	$-4e^{-1}$
$B = f''_{xy}$	$(-4xy+4x^3y)e^{-x^2-y^2}$	0	0
$C = f''_{yy}$	$(-2x^2+4x^2y^2)e^{-x^2-y^2}$	0	$-2e^{-1} < 0$
$AC-B^2$		0	$+$ lok. max. $= e^{-1}$

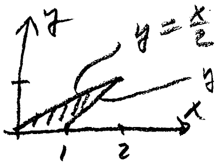
Eftersom $f(x,y) \geq 0$ $\forall (x,y)$ och $f(0,y) = 0$ så är $(0,y)$ lok. min. för alla y.

4. $\nabla f = (1,2) \neq \vec{0}$ så inga lokala extrempunkter.
$g(x,y) = x^2+xy+2y^2 = 7$ är en ellips vars axlar kan bestämmas med egenvärdesteori varefter man kan göra en ellipspolär parametrisering.
Vi använder i stället att $\nabla g = (2x+y, x+4y)$ är parallell med ∇f i extrempunkter, så där gäller

$x+4y = 2(2x+y) \iff 2y = 3x \iff y = \frac{3}{2}x$;

$7 = g(x, \frac{3}{2}x) = x^2(1+\frac{3}{2}+2\cdot\frac{9}{4}) = x^2\cdot 7 \iff x = \pm 1 \iff y = \pm\frac{3}{2}$;

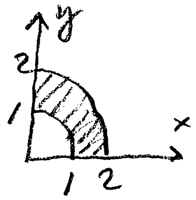
$f(1,\frac{3}{2}) = 4 = \max$, $f(-1,-\frac{3}{2}) = -4 = \min$.

5. 

$$\int_0^1\left(\int_{2y}^{y+1} \frac{x}{y+1}\,dx\right)dy = \frac{1}{2}\int_0^1 \left[\frac{x^2}{y+1}\right]_{2y}^{y+1} dy =$$

$$= \frac{1}{2}\int_0^1 (y+1)\,dy - 2\int_0^1 \frac{y^2}{y+1}\,dy = \frac{1}{2}\left[\frac{1}{2}y^2+y\right]_0^1 -$$

$$-2\int_0^1\left(y-1+\frac{1}{y+1}\right)dy = \frac{3}{4} - \left[y^2-2y+2\ln(y+1)\right]_0^1 = \frac{7}{4} - 2\ln 2.$$

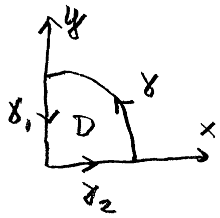
6. 

I polära koord. ges området av $1\le r\le 2,\ 0\le\varphi\le\frac{\pi}{2}$.

$$\iint_D \frac{1}{x^2} e^{-\frac{y}{x}}\,dx\,dy = \iint_{D'} \frac{1}{r^2\cos^2\varphi} e^{-\tan\varphi}\, r\,dr\,d\varphi =$$

$$= \int_1^2 \frac{1}{r}\,dr \int_0^{\frac{\pi}{2}} e^{-\tan\varphi}\cdot\frac{1}{\cos^2\varphi}\,d\varphi = \ln 2\cdot\left[-e^{-\tan\varphi}\right]_0^{\frac{\pi}{2}} =$$

$$= \ln 2\cdot\left(e^0 - \lim_{\varphi\to\frac{\pi}{2}-} e^{-\tan\varphi}\right) = \ln 2\cdot(1-0) = \ln 2.$$

7. 

$$\left(\int_\gamma + \int_{\gamma_1} + \int_{\gamma_2}\right)\left(\underbrace{(xy+e^{2x})}_{P}dx + \underbrace{(x^2+e^y)}_{Q}dy\right) =$$

$$= \iint_D (Q'_x - P'_y)\,dx\,dy = \int_0^1\left(\int_0^{\sqrt{1-y^2}} x\,dx\right)dy = \int_0^1\left[\frac{1}{2}x^2\right]_0^{\sqrt{1-y^2}} dy =$$

$$= \frac{1}{2}\int_0^1 (1-y^2)\,dy = \frac{1}{2}\left[y - \frac{1}{3}y^3\right]_0^1 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

$\int_{\gamma_1} P\,dx + Q\,dy = \int_1^0 e^y\,dy = 1-e$, $\int_{\gamma_2} P\,dx+Q\,dy = \int_0^1 e^{2x}\,dx = \frac{1}{2}(e^2-1)$.

Alltså $\int_\gamma P\,dx+Q\,dy = \frac{1}{3} - 1 + e - \frac{1}{2}e^2 + \frac{1}{2} = -\frac{1}{6} + e - \frac{1}{2}e^2$

8. Utanför origo är f C^1 enligt deriveringsregler och alltså differentierbar.

$f(x,0) = x \ \forall x \Rightarrow f'_x(x,0) = 1 \ \forall x$

$f(0,y) = -y \ \forall y \Rightarrow f'_y(0,y) = -1 \ \forall y$

$$\rho(h,k) = \frac{f(h,k)-f(0,0)-f'_x(0,0)h - f'_y(0,0)k}{\sqrt{h^2+k^2}} = \frac{\frac{h^3-k^3}{h^2+k^2} - h + k}{\sqrt{h^2+k^2}} =$$

$= \frac{h^2k - hk^2}{(h^2+k^2)^{3/2}} \not\to 0$ då $(h,k)\to(0,0)$, t.ex. $\rho(h,-h) = -\frac{1}{\sqrt{2}},\ h>0$,

Så f är ej differentierbar i origo.