LMA320 Flervariabelanalys 17 jan 06

1. $f(x,y,z) = z^3 - 6xyz + ax^3 + by^2 + c$

$\nabla f = (-6yz + 3ax^2, -6xz + 2by, 3z^2 - 6xy)$

Villkoren är $f(1,1,1) = f'_x(1,1,1) = f'_y(1,1,1) = 0$, dvs.

$$\begin{cases} -5 + a + b + c = 0 \\ -6 + 3a = 0 \\ -6 + 2b = 0 \end{cases} \iff \begin{cases} a = 2 \\ b = 3 \\ c = 0 \end{cases}$$

och då är $\nabla f(1,1,1) = (0,0,-3) \neq \vec{0}$ så ytan har tangentplan i punkten.

2. a) $$\begin{cases} u = \frac{x}{y} \\ v = xy \end{cases} \implies \begin{cases} u'_x = \frac{1}{y} & u'_y = -\frac{x}{y^2} \\ v'_x = y & v'_y = x \end{cases} \qquad x,y>0 \implies u,v>0$$

$$z'_x = z'_u u'_x + z'_v v'_x = \frac{1}{y} z'_u + y z'_v$$

$$z'_y = z'_u u'_y + z'_v v'_y = -\frac{x}{y^2} z'_u + x z'_v$$

(PDE) $z = x z'_x + y z'_y = \left(\frac{x}{y} - \frac{x}{y}\right) z'_u + (xy + xy) z'_v = 2v z'_v$

b) (PDE) $\iff z'_v - \frac{1}{2v} z = 0$; int. faktor $e^{\int -\frac{1}{2v} dv} = e^{-\frac{1}{2}\ln v} = \frac{1}{\sqrt{v}}$

så (PDE) $\iff \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{v}} z\right) = 0 \iff \frac{1}{\sqrt{v}} z = f(u)$, f kontinuerligt deriverbar.

$$\iff z = \sqrt{v}\, f(u) = \sqrt{xy}\, f\left(\frac{x}{y}\right)$$

3. $f(x,y) = \frac{x+y}{1+x^2+y^2}$

$$f'_x = \frac{1+x^2+y^2-(x+y)\cdot 2x}{(1+x^2+y^2)^2} = \frac{1-x^2-2xy+y^2}{(1+x^2+y^2)^2} = \frac{1-2xy-(x^2-y^2)}{(1+x^2+y^2)^2}$$

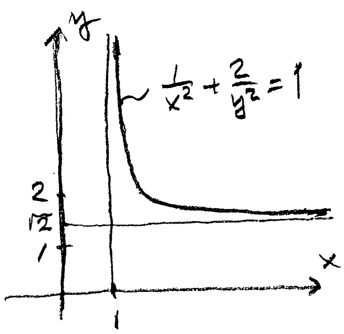
$$f'_y = \frac{1+x^2-2xy-y^2}{(1+x^2+y^2)^2} = \frac{1-2xy+(x^2-y^2)}{(1+x^2+y^2)^2}$$

så $f'_x = 0 = f'_y \iff \begin{cases} 1-2xy = 0 \\ x^2-y^2 = 0 \end{cases} \iff (x,y) = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{1+\frac{1}{2}+\frac{1}{2}} = \frac{1}{\sqrt{2}}, \qquad f\left(\tfrac{-1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}.$$

Eftersom $f(x,y) \to 0$ då $x^2+y^2 \to \infty$ är dessa f:s största resp. minsta värde.

4\.



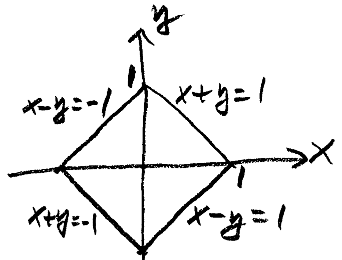
Bivillkoret är en kurva enl. fig. och på den kan $x \to \infty$ eller $y \to \infty$ samtidigt som $y \to \sqrt{2}+$ resp. $x \to 1+$; i båda fallen gäller $f(x,y) = xy \to \infty$.

f saknar alltså största värde under bivillkoret $g(x,y) = \frac{1}{x^2} + \frac{2}{y^2} = 1$. Däremot finns minsta värde i en punkt där $\nabla f = (y, x)$ är parallell med $\nabla g = \left(-\frac{2}{x^3}, -\frac{4}{y^3}\right)$ ($\nabla g \neq \vec{0}$ alltid).

$\nabla f \parallel \nabla g \iff y \cdot \frac{-4}{y^3} = x \cdot \frac{-2}{x^3} \iff 2x^2 = y^2$, vilket insatt i $g(x,y)=1$ ger $\frac{1}{x^2} + \frac{1}{x^2} = 1 \iff x^2 = 2 \implies y^2 = 4$

så $f_{min} = f(\sqrt{2}, 2) = 2\sqrt{2}$.

5\.



Subst $\begin{cases} u = x-y \\ v = x+y \end{cases}$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\iint_K \frac{(x+y)^2}{1+(x-y)^2}\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \frac{v^2}{1+u^2} \cdot \frac{1}{2}\,du\,dv =$$

$$= \frac{1}{2}\left[\frac{1}{3}v^3\right]_{-1}^{1}\left[\arctan u\right]_{-1}^{1} = \frac{1}{2}\cdot\frac{2}{3}\cdot 2\cdot\frac{\pi}{4} = \frac{\pi}{6}$$

6\. $$\iiint_D \frac{x}{1+y^2+z^2}\,dx\,dy\,dz = \iint_E \frac{1}{1+y^2+z^2}\int_{\sqrt{y^2+z^2}}^{\sqrt{2(y^2+z^2)}} x\,dx\,dy\,dz =$$

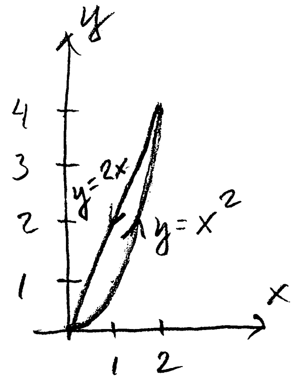
$$= \iint_E \frac{1}{1+y^2+z^2}\left[\frac{1}{2}x^2\right]_{\sqrt{y^2+z^2}}^{\sqrt{2(y^2+z^2)}} dy\,dz = \frac{1}{2}\iint_E \frac{y^2+z^2}{1+y^2+z^2}\,dy\,dz =$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{r^2}{1+r^2}\,r\,dr\,d\varphi = \pi\int_0^1\left(r - \frac{r}{1+r^2}\right)dr =$$

$$= \pi\left[\frac{1}{2}r^2 - \frac{1}{2}\ln(1+r^2)\right]_0^1 = \frac{\pi}{2}(1-\ln 2),$$

E = enhetscirkelskivan i yz-planet.

7.



Greens formel ger

$$\int_\gamma y^2\,dx + x^2\,dy = \int_0^2 \int_{x^2}^{2x} (2x-2y)\,dy\,dx =$$

$$= \int_0^2 \left[2xy - y^2\right]_{x^2}^{2x} dx =$$

$$= \int_0^2 (\underbrace{4x^2 - 4x^2}_{=0} - 2x^3 + x^4)\,dx =$$

$$= \left[-\tfrac{1}{2}x^4 + \tfrac{1}{5}x^5\right]_0^2 = -8 + \frac{32}{5} = -\frac{8}{5}.$$

Alt. Direkt uträkning. $\gamma = \gamma_1 + \gamma_2$, där

$$\gamma_1 : \begin{cases} x = x \\ y = x^2 \end{cases} \quad 0 \xrightarrow{x} 2 \qquad \gamma_2 : \begin{cases} x = x \\ y = 2x \end{cases} \quad 2 \xrightarrow{x} 0$$

Så integralen är $\int_0^2 (x^4 + x^2 \cdot 2x)\,dx + \int_2^0 (4x^2 + x^2 \cdot 2)\,dx =$

$$= \left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4\right]_0^2 + \left[2x^3\right]_2^0 = \frac{32}{5} + 8 - 16 = -\frac{8}{5}.$$