Matematik
INTEGRATIONSTEORI (5p)
(GU$[MAF440]$ **,CTH**$[TMV100]$**)**
Dag, tid, sal: 8 oktober 2004, fm, v
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Inlämning skall ske i uppgifternas ordning; v.g. sidnumrera!

1. Suppose

$$f_n(x) = n \mid x \mid e^{-\frac{nx^2}{2}},\ x \in \mathbf{R},\ n \in \mathbf{N}_+.$$

Show that there is no $g \in L^1(m)$ such that $f_n \leq g$ for all $n \in \mathbf{N}_+$.

2. Set

$$f(x) = \lim_{T\to\infty} \int_0^T \frac{\sin t}{x+t} dt,\ x \geq 0$$

and

$$g(x) = \frac{f(x)}{\sqrt{x}},\ x \geq 0.$$

Prove that g is Lebesgue integrable on $[0, \infty[$.

3. a) Let $\mathcal{M}$ be an algebra of subsets of X and $\mathcal{N}$ an algebra of subsets of Y. Furthermore, let S be the set of all finite unions of sets of the type $A \times B$, where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Prove that S is an algebra of subsets of $X \times Y$.

b) Assume $\mathcal{M}$ is a σ-algebra of subsets of X and $\mathcal{N}$ a σ-algebra of subsets of Y and let $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu)$ be a finite positive measure space. Prove that to each $E \in \mathcal{M} \otimes \mathcal{N}$ and $\varepsilon > 0$ there exists $F \in S$ such that

$$\mu(E \Delta F) < \varepsilon.$$

4. Formulate and prove the Fatous Lemma.

5. Let $\mathcal{C}$ be a collection of open balls and set $V = \cup_{B\in\mathcal{C}} B$. Prove that to each $c < m_n(V)$ there exist pairwise disjoint $B_1, ..., B_k \in \mathcal{C}$ such that

$$\Sigma_{i=1}^k m_n(B_i) > 3^{-n} c.$$