LÖSNINGAR
INTEGRATIONSTEORI (5p)
(**GU**[$MAF440$] ,**CTH**[$TMV100$])
Dag, tid: 8 oktober 2004 fm
Hjälpmedel: Inga.

1. Suppose

$$f_n(x) = n \mid x \mid e^{-\frac{nx^2}{2}},\ x \in \mathbf{R}, n \in \mathbf{N}_+.$$

Show that there is no $g \in L^1(m)$ such that $f_n \leq g$ for all $n \in \mathbf{N}_+$.

Solution. We have

$$\lim_{n\to\infty} f_n(x) = 0 \text{ all } x \in \mathbf{R}$$

and

$$\int_{-\infty}^{\infty} f_n(x)dx = \left[\sqrt{n}x = y\right] = \int_{-\infty}^{\infty} \mid y \mid e^{-\frac{y^2}{2}} dy \ = 2 \text{ all } n \in \mathbf{N}_+.$$

The Lebesgue Dominated Convergence Theorem now implies that there is no $g \in L^1(m)$ such that $\mid f_n \mid \leq g$ for all $n \in \mathbf{N}_+$. Since $f_n = \mid f_n \mid$ we are done.

2. Set

$$f(x) = \lim_{T\to\infty} \int_0^T \frac{\sin t}{x+t} dt,\ x \geq 0$$

and

$$g(x) = \frac{f(x)}{\sqrt{x}},\ x \geq 0.$$

Prove that g is Lebesgue integrable on $[0, \infty[$.

Solution. Let $x \geq 0$. By partial integration

$$\int_{\frac{\pi}{2}}^T \frac{\sin t}{x+t} dt = -\frac{\cos T}{x+T} - \int_{\frac{\pi}{2}}^T \frac{\cos t}{(x+t)^2} dt$$

and we get

$$f(x) = \int_0^{\frac{\pi}{2}} \frac{\sin t}{x+t} dt - \int_{\frac{\pi}{2}}^{\infty} \frac{\cos t}{(x+t)^2} dt.$$

Note that f is a Borel function by the Tonelli Theorem.

Now

$$| f(x) | \leq \int_0^{\frac{\pi}{2}} \frac{| \sin t |}{t} dt + \int_{\frac{\pi}{2}}^{\infty} \frac{1}{(x+t)^2} dt$$

and since $| \sin t | \leq t$ for $t \geq 0$, we get

$$| f(x) | \leq \frac{\pi}{2} + \frac{1}{x + \frac{\pi}{2}} \leq \frac{\pi}{2} + \frac{2}{\pi}.$$

Hence

$$\int_0^1 \frac{| f(x) |}{\sqrt{x}} dx < \infty.$$

Furthermore,

$$| f(x) | \leq \int_0^{\frac{\pi}{2}} \frac{1}{x} dt + \int_{\frac{\pi}{2}}^{\infty} \frac{1}{(x+t)^2} dt$$

$$= \frac{\pi}{2x} + \frac{1}{x + \frac{\pi}{2}} \leq (\frac{\pi}{2} + 1)\frac{1}{x}$$

and it follows that

$$\int_1^{\infty} \frac{| f(x) |}{\sqrt{x}} dx < \infty.$$

Summing up we conclude that g is Lebesgue integrable on $[0, \infty[$.

3. a) Let $\mathcal{M}$ be an algebra of subsets of X and $\mathcal{N}$ an algebra of subsets of Y. Furthermore, let S be the set of all finite unions of sets of the type $A \times B$, where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Prove that S is an algebra of subsets of $X \times Y$.

b) Assume $\mathcal{M}$ is a σ-algebra of subsets of X and $\mathcal{N}$ a σ-algebra of subsets of Y and let $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu)$ be a finite positive measure space. Prove that to each $E \in \mathcal{M} \otimes \mathcal{N}$ and $\varepsilon > 0$ there exists $F \in S$ such that

$$\mu(E \Delta F) < \varepsilon.$$

Solution. a) The main point in the proof is to show that S is closed under finite intersections. To see this let

$$E = \cup_{k=1}^{M}(A_k \times B_k)$$

and

$$F = \cup_{k=1}^{N}(C_k \times D_k)$$

where $A_1, ..., A_M,\ C_1, ..., C_N \in \mathcal{M}$ and $B_1, ..., B_M,\ D_1, ..., D_N \in \mathcal{N}$. It is enough to prove that $E \cap F \in S$. But

$$E \cap F = \cup_{\substack{1 \le i \le M \\ 1 \le j \le N}}((A_i \cap C_j) \times (B_i \cap D_j))$$

and we are done.

To prove that S is an algebra first note that $\phi \in S$ and that S is closed under finite unions. If E is as above it remains to prove that the complement E^c belongs to S. But

$$E^c = \cap_{k=1}^{M}(A_k \times B_k)^c$$

$$= \cap_{k=1}^{M}((A_k^c \times Y) \cup (X \times B_k^c))$$

and it follows $E^c \in S$.

b) Let Σ be the class of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the property in b) holds. Clearly, $\phi \in \Sigma$. Now let $E \in \Sigma$. If $F \in S$, then $F^c \in S$ and $E \Delta F = E^c \Delta F^c$. Hence $E^c \in \Sigma$.

Finally, let $E_i \in \Sigma$, $i \in \mathbf{N}_+$. We shall prove that $E = \cup_{i=1}^{\infty} E_i \in \Sigma$. To this end let $\varepsilon > 0$ be arbitrary and choose $F_i \in S$ such that

$$\mu(E_i \Delta F_i) < 2^{-i}\varepsilon$$

for all $i \in \mathbf{N}_+$. Since

$$E\Delta(\cup_{i=1}^{\infty} F_i) \subseteq \cup_{i=1}^{\infty} E_i \Delta F_i,$$

$$\mu(E\Delta(\cup_{i=1}^{\infty} F_i) \le \Sigma_{i=1}^{\infty}\mu(E_i \Delta F_i) < \varepsilon.$$

Now

$$E\Delta(\cup_{i=1}^{\infty} F_i) = (\cap_{i=1}^{\infty}(E \cap F_i^c)) \cup (E^c \cap (\cup_{i=1}^{\infty} F_i))$$

and since μ is a finite positive measure it follows that

$$\mu((\cap_{i=1}^{n}(E \cap F_i^c)) \cup (E^c \cap (\cup_{i=1}^{\infty} F_i))) < \varepsilon$$

if n is sufficiently large. Hence

$$\mu(E\Delta(\cup_{i=1}^n F_i)) \leq \mu((\cap_{i=1}^n (E \cap F_i^c)) \cup (E^c \cap (\cup_{i=1}^n F_i))) < \varepsilon$$

if n is large, which proves that $\cup_{i=1}^\infty E_i \in \Sigma$. Thus Σ is a σ-algebra contained in $\mathcal{M} \otimes \mathcal{N}$ and since Σ contains all measurable rectangles $\Sigma = \mathcal{M} \otimes \mathcal{N}$.

4. Formulate and prove the Fatous Lemma.

5. Let $\mathcal{C}$ be a collection of open balls and set $V = \cup_{B \in \mathcal{C}} B$. Prove that to each $c < m_n(V)$ there exist pairwise disjoint $B_1, ..., B_k \in \mathcal{C}$ such that

$$\Sigma_{i=1}^k m_n(B_i) > 3^{-n} c.$$