LÖSNINGAR
INTEGRATIONSTEORI (5p)
(GU$[MAF440]$,**CTH**$[TMV100]$**)**
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Hjälpmedel: Inga.
Skrivtid: 4 timmar

1. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $(Y, \mathcal{N})$ a measurable space. Furthermore, suppose $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and let $\nu = \mu f^{-1}$, that is $\nu(B) = \mu(f^{-1}(B))$ if $B \in \mathcal{N}$. Show that f is $(\mathcal{M}^-, \mathcal{N}^-)$-measurable, where $\mathcal{M}^-$ denotes the completion of $\mathcal{M}$ with respect to μ and $\mathcal{N}^-$ the completion of $\mathcal{N}$ with respect to ν.

Solution: Suppose $B \in \mathcal{N}^-$. We will prove that $f^{-1}(B) \in \mathcal{M}^-$. To this end, choose $B_0, B_1 \in \mathcal{N}$ such that $B_0 \subseteq B \subseteq B_1$ and $\nu(B_1 \setminus B_0) = 0$. Then $f^{-1}(B_0)$, $f^{-1}(B_1) \in \mathcal{M}$ and $f^{-1}(B_0) \subseteq f^{-1}(B) \subseteq f^{-1}(B_1)$. Furthermore, $f^{-1}(B_1) \setminus f^{-1}(B_0) = f^{-1}(B_1 \setminus B_0)$ and we get

$$\mu(f^{-1}(B_1) \setminus f^{-1}(B_0)) = \nu(B_1 \setminus B_0) = 0.$$

Thus $f^{-1}(B) \in \mathcal{M}^-$ and we are done.

2. Compute the following limit and justify the calculations:

$$\lim_{n\to\infty} \int_0^\infty (1 + \frac{x}{n})^{n^2} e^{-nx} dx.$$

Solution. We have

$$(1 + \frac{x}{n})^{n^2} e^{-nx} = e^{n^2 \ln(1+\frac{x}{n}) - nx}$$
$$= e^{n^2(\frac{x}{n} - \frac{x^2}{2n^2} + (\frac{x}{n})^3 B(\frac{x}{n})) - nx}$$

where B is bounded in a neighbourhood of the origin. Accordingly from this,

$$\lim_{n\to\infty} (1 + \frac{x}{n})^{n^2} e^{-nx} = e^{-\frac{x^2}{2}}.$$

To find a majorant, let $x \geq 0$ be fixed and introduce the function

$$f(n) = n^2 \ln(1 + \frac{x}{n}) - nx$$

defined for all real $n \geq 1$. We claim that

$$f'(n) = 2n \ln(1 + \frac{x}{n}) - \frac{2x + \frac{x^2}{n}}{1 + \frac{x}{n}} \leq 0.$$

To see this put

$$g(t) = 2(1+t)\ln(1+t) - (2t + t^2) \text{ for } t \geq 0$$

and note that $f'(n) \leq 0$ if and only if $g(\frac{x}{n}) \leq 0$. But $g(0) = 0$ and

$$g'(t) = 2(\ln(1+t) - t) \leq 0$$

and it follows that $g \leq 0$. Thus $f'(n) \leq 0$ and, hence, $f(n) \leq f(1)$. Now

$$(1 + \frac{x}{n})^{n^2} e^{-nx} \leq (1+x)e^{-x} \in L^1(m_{0,\infty})$$

where $m_{0,\infty}$ is Lebesgue measure on $[0, \infty[$ and the Lebesgue Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_0^\infty (1 + \frac{x}{n})^{n^2} e^{-nx} dx = \int_0^\infty e^{-\frac{x^2}{2}} dx = \sqrt{\frac{\pi}{2}}.$$

3. Suppose $a > 0$ and

$$\mu_a = e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} \delta_n$$

where δ_n is the Dirac measure on $\mathbf{N} = \{0, 1, 2, ...\}$ at the point $n \in \mathbf{N}$, that is $\delta_n(A) = \chi_A(n)$ if $n \in \mathbf{N}$ and $A \subseteq \mathbf{N}$. Prove that

$$(\mu_a \times \mu_b)s^{-1} = \mu_{a+b}$$

for all $a, b > 0$ if $s(x, y) = x + y$, $x, y \in \mathbf{N}$.

Solution. If μ and ν are finite positive measures on $\mathbf{N}$, we define $\mu * \nu = (\mu \times \nu)s^{-1}$. Now, given $a, b > 0$ and $A \in \mathcal{P}(\mathbf{N})$, the Tonelli Theorem implies that

$$(\mu_a * \mu_b)(A) = (\mu_a \times \mu_b)(\{(x,y) \in \mathbf{N}^2;\ x + y \in A\})$$
$$= \int_{\mathbf{N}} \mu_a(\{x \in \mathbf{N};\ x + y \in A\}) d\mu_b(y)$$

and by applying the Lebesgue Monotone Convergence Theorem we have,

$$(\mu_a * \mu_b)(A) = \sum_{i=0}^{\infty} e^{-a} \frac{a^i}{i!} \int_{\mathbf{N}} \delta_i(\{x \in \mathbf{N};\ x + y \in A\}) d\mu_b(y)$$

$$= \sum_{i=0}^{\infty} e^{-a} \frac{a^i}{i!} (\delta_i * \mu_b)(A).$$

In a similar way,

$$(\delta_i * \mu_b)(A) = \sum_{j=0}^{\infty} e^{-b} \frac{b^j}{j!} (\delta_i * \delta_j)(A).$$

Since $\delta_i * \delta_j = \delta_{i+j}$, we get

$$(\mu_a * \mu_b)(A) = \sum_{i,j=0}^{\infty} e^{-(a+b)} \frac{a^i b^j}{i! j!} \delta_{i+j}(A)$$

$$= \sum_{n=0}^{\infty} (e^{-(a+b)} \delta_n(A) \sum_{\substack{i+j=n \\ i.j \geq 0}} \frac{a^i b^j}{i! j!}) = \sum_{n=0}^{\infty} e^{-(a+b)} \frac{(a+b)^n}{n!} \delta_n(A) = \mu_{a+b}(A).$$

4. Suppose $f :]a, b[\times X \to \mathbf{R}$ is a function such that $f(t, \cdot) \in \mathcal{L}^1(\mu)$ for each $t \in]a, b[$ and, moreover, assume $\frac{\partial f}{\partial t}$ exists and

$$\mid \frac{\partial f}{\partial t}(t, x) \mid \leq g(x) \text{ for all } (t, x) \in]a, b[\times X$$

where $g \in \mathcal{L}^1(\mu)$. Set

$$F(t) = \int_X f(t, x) d\mu(x) \text{ if } t \in]a, b[\,.$$

Prove that F is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(t,x)d\mu(x) \text{ if } t \in \left]a,b\right[.$$

5. Suppose θ is an outer measure on X and let $\mathcal{M}(\theta)$ be the set of all $A \subseteq X$ such that

$$\theta(E) = \theta(E \cap A) + \theta(E \cap A^c) \text{ for all } E \subseteq X.$$

Prove that $\mathcal{M}(\theta)$ is a σ-algebra and that the restriction of θ to $\mathcal{M}(\theta)$ is a complete measure.