# CHAPTER 1

# MEASURES

## Introduction

The Riemann integral, dealt with in calculus courses, is well suited for computations but less suited for dealing with limit processes. In this course we will introduce the so called Lebesgue integral, which keeps the advantages of the Riemann integral and eliminates its drawbacks. At the same time we will develop a general measure theory which serves as the basis of contemporary analysis and probability.

In this introductory chapter we set forth some basic concepts of measure theory, which will open for abstract Lebesgue integration.

## 1.1. $\sigma$-Algebras and Measures

Throughout this course

$\mathbf{N} = \{0, 1, 2, ...\}$ (the set of natural numbers)
$\mathbf{Z} = \{..., -2, -1, 0, 1, , 2, ...\}$ (the set of integers)
$\mathbf{Q} =$ the set of rational numbers
$\mathbf{R} =$ the set of real numbers
$\mathbf{C} =$ the set of complex numbers.

If $A \subseteq \mathbf{R}$, $A_+$ is the set of all strictly positive elements in $A$.

If $f$ is a function of $A$ into $B$, this means that to every $x \in A$ there corresponds a point $f(x) \in B$ and we write $f : A \to B$. A function is often called a map or a mapping. The function $f$ is injective if

$$(x \neq y) \Rightarrow (f(x) \neq f(y))$$

and surjective if to each $y \in B$, there exists an $x \in A$ such that $f(x) = y$. An injective and surjective function is said to be bijective.

A set $A$ is finite if either $A$ is empty or there exist an $n \in \mathbf{N}_+$ and a bijection $f : \{1, ..., n\} \to A$. The empty set is denoted by $\phi$. A set $A$ is said to be denumerable if there exists a bijection $f : \mathbf{N}_+ \to A$. A subset of a denumerable set is said to be at most denumerable.

Let $X$ be a set. For any $A \subseteq X$, the indicator function $\chi_A$ of $A$ relative to $X$ is defined by the equation

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in A^c. \end{cases}$$

The indicator function $\chi_A$ is sometimes written $1_A$. We have the following relations:

$$\chi_{A^c} = 1 - \chi_A$$

$$\chi_{A \cap B} = \min(\chi_A, \chi_B) = \chi_A \chi_B$$

and

$$\chi_{A \cup B} = \max(\chi_A, \chi_B) = \chi_A + \chi_B - \chi_A \chi_B.$$

**Definition 1.1.1.** Let $X$ be a set.

a) A collection $\mathcal{A}$ of subsets of $X$ is said to be an algebra in $X$ if $\mathcal{A}$ has the following properties:

(i) $X \in \mathcal{A}$.
(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, where $A^c$ is the complement of $A$ relative to $X$.
(iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

(b) A collection $\mathcal{M}$ of subsets of $X$ is said to be a $\sigma$-algebra in $X$ if $\mathcal{M}$ is an algebra with the following property:

If $A_n \in \mathcal{M}$ for all $n \in \mathbf{N}_+$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$.

If $\mathcal{M}$ is a $\sigma$-algebra in $X$, $(X, \mathcal{M})$ is called a measurable space and the members of $\mathcal{M}$ are called measurable sets. The so called power set $\mathcal{P}(X)$, that is the collection of all subsets of $X$, is a $\sigma$-algebra in $X$. It is simple to prove that the intersection of any family of $\sigma$-algebras in $X$ is a $\sigma$-algebra. It follows that if $\mathcal{E}$ is any subset of $\mathcal{P}(X)$, there is a unique smallest $\sigma$-algebra $\sigma(\mathcal{E})$ containing $\mathcal{E}$, namely the intersection of all $\sigma$-algebras containing $\mathcal{E}$.

The $\sigma$-algebra $\sigma(\mathcal{E})$ is called the $\sigma$-algebra generated by $\mathcal{E}$. The $\sigma$-algebra generated by all open intervals in $\mathbf{R}$ is denoted by $\mathcal{R}$. It is readily seen that the $\sigma$-algebra $\mathcal{R}$ contains every subinterval of $\mathbf{R}$. Be fore we proceed, recall that a subset $E$ of $\mathbf{R}$ is open if to each $x \in E$ there exists an open subinterval of $\mathbf{R}$ contained in $E$ and containing $x$; the complement of an open set is said to be closed. We claim that $\mathcal{R}$ contains every open $U$ subset of $\mathbf{R}$. To see this suppose $x \in U$ and let $x \in \,]a, b[\, \subseteq U$, where $-\infty < a < b < \infty$. Now pick $r, s \in \mathbf{Q}$ such that $a < r < x < s < b$. Then $x \in \,]r, s[\, \subseteq U$ and it follows that $U$ is the union of all bounded open intervals with rational boundary points contained in $U$. Since this family of intervals is at most denumberable we conclude that $U \in \mathcal{R}$. In addition, any closed set belongs to $\mathcal{R}$ since its complements is open. It is by no means simple to grasp the definition of $\mathcal{R}$ at this stage but the reader will successively see that the $\sigma$-algebra $\mathcal{R}$ has very nice properties. At the very end of Section 1.3, using the so called Axiom of Choice, we will exemplify a subset of the real line which does not belong to $\mathcal{R}$. In fact, an example of this type can be constructed without the Axiom of Choice (see Dudley's book $[D]$).

In measure theory, inevitably one encounters $\infty$. For example the real line has infinite length. Below $[0, \infty] = [0, \infty[ \cup \{\infty\}$ . The inequalities $x \leq y$ and $x < y$ have their usual meanings if $x, y \in [0, \infty[$. Furthermore, $x \leq \infty$ if $x \in [0, \infty]$ and $x < \infty$ if $x \in [0, \infty[$ . We define $x + \infty = \infty + x = \infty$ if $x, y \in [0, \infty]$ , and

$$x \cdot \infty = \infty \cdot x = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } 0 < x \leq \infty. \end{cases}$$

Sums and multiplications of real numbers are defined in the usual way.

If $A_n \subseteq X$, $n \in \mathbf{N}_+$, and $A_k \cap A_n = \phi$ if $k \neq n$, the sequence $(A_n)_{n \in \mathbf{N}_+}$ is called a disjoint denumerable collection. If $(X, \mathcal{M})$ is a measurable space, the collection is called a denumerable measurable partition of $A$ if $A = \cup_{n=1}^{\infty} A_n$ and $A_n \in \mathcal{M}$ for every $n \in \mathbf{N}_+$. Some authors call a denumerable collection of sets a countable collection of sets.

**Definition 1.1.2.** (a) Let $\mathcal{A}$ be an algebra of subsets of $X$. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a content if

(i) $\mu(\phi) = 0$
(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{A}$ and $A \cap B = \phi$.

(b) If $(X, \mathcal{M})$ is a measurable space a content $\mu$ defined on the $\sigma$-algebra $\mathcal{M}$ is called a positive measure if it has the following property:

For any disjoint denumerable collection $(A_n)_{n \in \mathbf{N}_+}$ of members of $\mathcal{M}$

$$\mu(\cup_{n=1}^{\infty} A_n) = \Sigma_{n=1}^{\infty} \mu(A_n).$$

If $(X, \mathcal{M})$ is a measurable space and the function $\mu : \mathcal{M} \to [0, \infty]$ is a positive measure, $(X, \mathcal{M}, \mu)$ is called a positive measure space. The quantity $\mu(A)$ is called the $\mu$-measure of $A$ or simply the measure of $A$ if there is no ambiguity. Here $(X, \mathcal{M}, \mu)$ is called a probability space if $\mu(X) = 1$, a finite positive measure space if $\mu(X) < \infty$, and a $\sigma$-finite positive measure space if $X$ is a denumerable union of measurable sets with finite $\mu$-measure. The measure $\mu$ is called a probability measure, finite measure, and $\sigma$-finite measure, if $(X, \mathcal{M}, \mu)$ is a probability space, a finite positive measure space, and a $\sigma$-finite positive measure space, respectively. A probability space is often denoted by $(\Omega, \mathcal{F}, P)$. A member $A$ of $\mathcal{F}$ is called an event.

As soon as we have a positive measure space $(X, \mathcal{M}, \mu)$, it turns out to be a fairly simple task to define a so called $\mu$-integral

$$\int_X f(x) d\mu(x)$$

as will be seen in Chapter 2.

The class of all finite unions of subintervals of $\mathbf{R}$ is an algebra which is denoted by $\mathcal{R}_0$. If $A \in \mathcal{R}_0$ we denote by $l(A)$ the Riemann integral

$$\int_{-\infty}^{\infty} \chi_A(x)dx$$

and it follows from courses in calculus that the function $l : \mathcal{R}_0 \to [0, \infty]$ is a content. The algebra $\mathcal{R}_0$ is called the Riemann algebra and $l$ the Riemann content. If $I$ is a subinterval of $\mathbf{R}$, $l(I)$ is called the length of $I$. Below we follow the convention that the empty set is an interval.

If $A \in \mathcal{P}(X)$, $c_X(A)$ equals the number of elements in $A$, when $A$ is a finite set, and $c_X(A) = \infty$ otherwise. Clearly, $c_X$ is a positive measure. The measure $c_X$ is called the counting measure on $X$.

Given $a \in X$, the probability measure $\delta_a$ defined by the equation $\delta_a(A) = \chi_A(a)$, if $A \in \mathcal{P}(X)$, is called the Dirac measure at the point $a$. Sometimes we write $\delta_a = \delta_{X,a}$ to emphasize the set $X$.

If $\mu$ and $\nu$ are positive measures defined on the same $\sigma$-algebra $\mathcal{M}$, the sum $\mu + \nu$ is a positive measure on $\mathcal{M}$. More generally, $\alpha\mu + \beta\nu$ is a positive measure for all real $\alpha$, $\beta \geq 0$. Furthermore, if $E \in \mathcal{M}$, the function $\lambda(A) = \mu(A \cap E)$, $A \in \mathcal{M}$, is a positive measure. Below this measure $\lambda$ will be denoted by $\mu^E$ and we say that $\mu^E$ is concentrated on $E$. If $E \in \mathcal{M}$, the class $\mathcal{M}_E = \{A \in \mathcal{M};\ A \subseteq E\}$ is a $\sigma$-algebra of subsets of $E$ and the function $\theta(A) = \mu(A)$, $A \in \mathcal{M}_E$, is a positive measure. Below this measure $\theta$ will be denoted by $\mu_{|E}$ and is called the restriction of $\mu$ to $\mathcal{M}_E$.

Let $I_1, ..., I_n$ be subintervals of the real line. The set

$$I_1 \times ... \times I_n = \{(x_1, ..., x_n) \in \mathbf{R}^n;\ x_k \in I_k,\ k = 1, ..., n\}$$

is called an $n$-cell in $\mathbf{R}^n$; its volume $\mathrm{vol}(I_1 \times ... \times I_n)$ is, by definition, equal to

$$\mathrm{vol}(I_1 \times ... \times I_n) = \Pi_{k=1}^n l(I_k).$$

If $I_1, ..., I_n$ are open subintervals of the real line, the $n$-cell $I_1 \times ... \times I_n$ is called an open $n$-cell. The $\sigma$-algebra generated by all open $n$-cells in $\mathbf{R}^n$ is denoted by $\mathcal{R}_n$. In particular, $\mathcal{R}_1 = \mathcal{R}$. A basic theorem in measure theory states that there exists a unique positive measure $v_n$ defined on $\mathcal{R}_n$ such that the measure of any $n$-cell is equal to its volume. The measure $v_n$ is called the volume measure on $\mathcal{R}_n$ or the volume measure on $\mathbf{R}^n$. Clearly, $v_n$ is $\sigma$-finite. The measure $v_2$ is called the area measure on $\mathbf{R}^2$ and $v_1$ the linear measure on $\mathbf{R}$.

**Theorem 1.1.1.** *The volume measure on* $\mathbf{R}^n$ *exists.*

Theorem 1.1.1 will be proved in Section 1.5 in the special case $n = 1$. The general case then follows from the existence of product measures in Section 3.4. An alternative proof of Theorem 1.1.1 will be given in Section 3.2. As soon as the existence of volume measure is established a variety of interesting measures can be introduced.

Next we prove some results of general interest for positive measures.

**Theorem 1.1.2.** *Let* $\mathcal{A}$ *be an algebra of subsets of* $X$ *and* $\mu$ *a content defined on* $\mathcal{A}$. *Then,*

(a) $\mu$ *is finitely additive, that is*

$$\mu(A_1 \cup ... \cup A_n) = \mu(A_1) + ... + \mu(A_n)$$

*if* $A_1, ..., A_n$ *are pairwise disjoint members of* $\mathcal{A}$.

(b) *if* $A, B \in \mathcal{A}$,

$$\mu(A) = \mu(A \setminus B) + \mu(A \cap B).$$

*Moreover, if* $\mu(A \cap B) < \infty$, *then*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

(c) $A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$ *if* $A, B \in \mathcal{A}$.

(d) $\mu$ *finitely sub-additive, that is*

$$\mu(A_1 \cup ... \cup A_n) \leq \mu(A_1) + ... + \mu(A_n)$$

*if* $A_1, ..., A_n$ *are members of* $\mathcal{A}$.

*If* $(X, \mathcal{M}, \mu)$ *is a positive measure space*

(e) $\mu(A_n) \to \mu(A)$ *if* $A = \cup_{n\in\mathbf{N}_+} A_n,\ A_n \in \mathcal{M}$, *and*

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \ .$$

(f) $\mu(A_n) \to \mu(A)$ *if* $A = \cap_{n\in\mathbf{N}_+} A_n,\ A_n \in \mathcal{M}$,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$$

*and* $\mu(A_1) < \infty$.
(g) $\mu$ *is sub-additive, that is for any denumerable collection* $(A_n)_{n\in\mathbf{N}_+}$ *of members of* $\mathcal{M}$,

$$\mu(\cup_{n=1}^{\infty} A_n) \leq \Sigma_{n=1}^{\infty} \mu(A_n).$$

PROOF (a) If $A_1, ..., A_n$ are pairwise disjoint members of $\mathcal{A}$,

$$\mu(\cup_{k=1}^{n} A_k) = \mu(A_1 \cup (\cup_{k=2}^{n} A_k))$$

$$= \mu(A_1) + \mu(\cup_{k=2}^{n} A_k)$$

and, by induction, we conclude that $\mu$ is finitely additive.

(b) Recall that

$$A \setminus B = A \cap B^c.$$

Now $A = (A \setminus B) \cup (A \cap B)$ and we get

$$\mu(A) = \mu(A \setminus B) + \mu(A \cap B).$$

Moreover, since $A \cup B = (A \setminus B) \cup B$,

$$\mu(A \cup B) = \mu(A \setminus B) + \mu(B)$$

and, if $\mu(A \cap B) < \infty$, we have

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(c) Part (b) yields $\mu(B) = \mu(B \setminus A) + \mu(A \cap B) = \mu(B \setminus A) + \mu(A)$, where the last member does not fall below $\mu(A)$.

(d) If $(A_i)_{i=1}^n$is a sequence of members of $\mathcal{A}$ define the so called disjunction $(B_k)_{k=1}^n$ of the sequence $(A_i)_{i=1}^n$ as

$$B_1 = A_1 \text{ and } B_k = A_k \setminus \cup_{i=1}^{k-1} A_i \text{ for } 2 \leq k \leq n.$$

Then $B_k \subseteq A_k$, $\cup_{i=1}^k A_i = \cup_{i=1}^k B_i$, $k = 1, .., n$, and $B_i \cap B_j = \phi$ if $i \neq j$. Hence, by Parts (a) and (c),

$$\mu(\cup_{k=1}^n A_k) = \Sigma_{k=1}^n \mu(B_k) \leq \Sigma_{k=1}^n \mu(A_k).$$

(e) Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then $A_n = B_1 \cup .... \cup B_n$, $B_i \cap B_j = \phi$ if $i \neq j$ and $A = \cup_{k=1}^\infty B_k$. Hence

$$\mu(A_n) = \Sigma_{k=1}^n \mu(B_k)$$

and

$$\mu(A) = \Sigma_{k=1}^\infty \mu(B_k).$$

Now e) follows, by the definition of the sum of an infinite series.

(f) Put $C_n = A_1 \setminus A_n$, $n \geq 1$. Then $C_1 \subseteq C_2 \subseteq C_3 \subseteq ...$,

$$A_1 \setminus A = \cup_{n=1}^\infty C_n$$

and $\mu(A) \leq \mu(A_n) \leq \mu(A_1) < \infty$. Thus

$$\mu(C_n) = \mu(A_1) - \mu(A_n)$$

and Part (e) shows that

$$\mu(A_1) - \mu(A) = \mu(A_1 \setminus A) = \lim_{n\to\infty} \mu(C_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n).$$

This proves (f).

(g) The result follows from Parts d) and e).

This completes the proof of Theorem 1.1.2.

The hypothesis "$\mu(A_1) < \infty$ " in Theorem 1.1.2 ( f) is not superfluous. If $c_{\mathbf{N}_+}$ is the counting measure on $\mathbf{N}_+$ and $A_n = \{n, n+1, ...\}$ , then $c_{\mathbf{N}_+}(A_n) = \infty$ for all $n$ but $A_1 \supseteq A_2 \supseteq ....$ and $c_{\mathbf{N}_+}(\cap_{n=1}^{\infty} A_n) = 0$ since $\cap_{n=1}^{\infty} A_n = \phi$.

If $A, B \subseteq X$, the symmetric difference $A\Delta B$ is defined by the equation

$$A\Delta B =_{def} (A \setminus B) \cup (B \setminus A).$$

Note that

$$\chi_{A\Delta B} =\mid \chi_A - \chi_B \mid .$$

Moreover, we have

$$A\Delta B = A^c \Delta B^c$$

and

$$(\cup_{i=1}^{\infty} A_i)\Delta(\cup_{i=1}^{\infty} B_i) \subseteq \cup_{i=1}^{\infty}(A_i \Delta B_i).$$

**Example 1.1.1.** Let $\mu$ be a finite positive measure on $\mathcal{R}$. We claim that to each set $E \in \mathcal{R}$ and $\varepsilon > 0$, there exists a set $A$, which is finite union of intervals (that is, $A$ belongs to the Riemann algebra $\mathcal{R}_0$), such that

$$\mu(E\Delta A) < \varepsilon.$$

To see this let $\mathcal{S}$ be the class of all sets $E \in \mathcal{R}$ for which the conclusion is true. Clearly $\phi \in \mathcal{S}$ and, moreover, $\mathcal{R}_0 \subseteq \mathcal{S}$. If $A \in \mathcal{R}_0$, $A^c \in \mathcal{R}_0$ and therefore $E^c \in \mathcal{S}$ if $E \in \mathcal{S}$. Now suppose $E_i \in \mathcal{S}$, $i \in \mathbf{N}_+$. Then to each $\varepsilon > 0$ and $i$ there is a set $A_i \in \mathcal{R}_0$ such that $\mu(E_i \Delta A_i) < 2^{-i}\varepsilon$. If we set

$$E = \cup_{i=1}^{\infty} E_i$$

then

$$\mu(E\Delta(\cup_{i=1}^{\infty} A_i)) \leq \Sigma_{i=1}^{\infty} \mu(E_i \Delta A_i) < \varepsilon.$$

Here

$$E\Delta(\cup_{i=1}^{\infty} A_i) = \{E \cap (\cap_{i=1}^{\infty} A_i^c)\} \cup \{E^c \cap (\cup_{i=1}^{\infty} A_i)\}$$

and Theorem 1.1.2 (f) gives that

$$\mu(\{E \cap (\cap_{i=1}^{n} A_i^c)\} \cup \{(E^c \cap (\cup_{i=1}^{\infty} A_i)\}) < \varepsilon$$

if $n$ is large enough (hint: $\cap_{i\in I}(D_i \cup F) = (\cap_{i\in I} D_i) \cup F$). But then

$$\mu(E\Delta \cup_{i=1}^{n} A_i) = \mu(\{E \cap (\cap_{i=1}^{n} A_i^c)\} \cup \{E^c \cap (\cup_{i=1}^{n} A_i)\}) < \varepsilon$$

if $n$ is large enough we conclude that the set $E \in \mathcal{S}$. Thus $\mathcal{S}$ is a $\sigma$-algebra and since $\mathcal{R}_0 \subseteq \mathcal{S} \subseteq \mathcal{R}$ it follows that $\mathcal{S} = \mathcal{R}$.

**Exercises**

1. Prove that the sets $\mathbf{N} \times \mathbf{N} = \{(i,j);\ i,j \in \mathbf{N}\}$ and $\mathbf{Q}$ are denumerable.

2. Suppose $\mathcal{A}$ is an algebra of subsets of $X$ and $\mu$ and $\nu$ two contents on $\mathcal{A}$ such that $\mu \leq \nu$ and $\mu(X) = \nu(X) < \infty$. Prove that $\mu = \nu$.

3. Suppose $\mathcal{A}$ is an algebra of subsets of $X$ and $\mu$ a content on $\mathcal{A}$ with $\mu(X) < \infty$. Show that

$$\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C)$$

$$-\mu(A \cap B) - \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C).$$

4. A collection $\mathcal{C}$ of subsets of $X$ is an algebra with the following property: If $A_n \in \mathcal{C}$, $n \in \mathbf{N}_+$ and $A_k \cap A_n = \phi$ if $k \neq n$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{C}$. Prove that $\mathcal{C}$ is a $\sigma$-algebra.

5. Let $(X, \mathcal{M})$ be a measurable space and $(\mu_k)_{k=1}^{\infty}$ a sequence of positive measures on $\mathcal{M}$ such that $\mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$ . Prove that the set function

$$\mu(A) = \lim_{k \to \infty} \mu_k(A),\ A \in \mathcal{M}$$

is a positive measure.

6. Let $(X, \mathcal{M}, \mu)$ be a positive measure space. Show that

$$\mu(\cap_{k=1}^{n} A_k) \leq \sqrt[n]{\Pi_{k=1}^{n} \mu(A_k)}$$

for all $A_1, ..., A_n \in \mathcal{M}$.

7. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space with $\mu(X) = \infty$. Show that for any $r \in [0, \infty[$ there is some $A \in \mathcal{M}$ with $r < \mu(A) < \infty$.

8. Show that the symmetric difference of sets is associative:

$$A\Delta(B\Delta C) = (A\Delta B)\Delta C.$$

9. $(X, \mathcal{M}, \mu)$ is a finite positive measure space. Prove that

$$\mid \mu(A) - \mu(B) \mid \leq \mu(A\Delta B).$$

10. Let $E = 2\mathbf{N}$. Prove that

$$c_{\mathbf{N}}(E\Delta A) = \infty$$

if $A$ is a finite union of intervals.

11. Suppose $(X, \mathcal{P}(X), \mu)$ is a finite positive measure space such that $\mu(\{x\}) > 0$ for every $x \in X$. Set

$$d(A, B) = \mu(A\Delta B),\ \ A, B \in \mathcal{P}(X).$$

Prove that

$$d(A, B) = 0 \ \Leftrightarrow A = B,$$
$$d(A, B) = d(B, A)$$

and

$$d(A, B) \leq d(A, C) + d(C, B).$$

12. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space. Prove that

$$\mu(\cup_{i=1}^{n} A_i) \geq \Sigma_{i=1}^{n} \mu(A_i) - \Sigma_{1 \leq i < j \leq n} \mu(A_i \cap A_j)$$

for all $A_1, ..., A_n \in \mathcal{M}$ and integers $n \geq 2$.

## 1.2. Measure Determining Classes

Suppose $\mu$ and $\nu$ are probability measures defined on the same $\sigma$-algebra $\mathcal{M}$, which is generated by a class $\mathcal{E}$. If $\mu$ and $\nu$ agree on $\mathcal{E}$, is it then true that $\mu$ and $\nu$ agree on $\mathcal{M}$? The answer is in general no. To show this, let

$$X = \{1, 2, 3, 4\}$$

and

$$\mathcal{E} = \{\{1, 2\}, \{1, 3\}\}.$$

Then $\sigma(\mathcal{E}) = \mathcal{P}(X)$. If $\mu = \frac{1}{4}c_X$ and

$$\nu = \frac{1}{6}\delta_{X,1} + \frac{1}{3}\delta_{X,2} + \frac{1}{3}\delta_{X,3} + \frac{1}{6}\delta_{X,4}$$

then $\mu = \nu$ on $\mathcal{E}$ and $\mu \neq \nu$.

In this section we will prove a basic result on measure determining classes for $\sigma$-finite measures. In this context we will introduce so called $\pi$-systems and $\lambda$-systems, which will also be of great value later in connection with the construction of so called product measures in Chapter 3.

**Definition 1.2.1.** A class $\mathcal{G}$ of subsets of $X$ is a $\pi$-system if $A \cap B \in \mathcal{G}$ for all $A, B \in \mathcal{G}$.

The class of all open $n$-cells in $\mathbf{R}^n$ is a $\pi$-system.

**Definition 1.2.2.** A class $\mathcal{D}$ of subsets of $X$ is a $\lambda$-system if the following properties hold:

(a) $X \in \mathcal{D}$.

(b) If $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.

(c) If $(A_n)_{n\in\mathbf{N}_+}$ is a disjoint denumerable collection of members of the class $\mathcal{D}$, then $\cup_{n=1}^{\infty}A_n \in \mathcal{D}$.

**Theorem 1.2.1.** *If a class $\mathcal{M}$ is both a $\pi$-system and $\lambda$-system, then $\mathcal{M}$ is a $\sigma$-algebra.*

PROOF. If $A \in \mathcal{M}$, then $A^c = X \setminus A \in \mathcal{M}$ since $X \in \mathcal{M}$ and $\mathcal{M}$ is a $\lambda$-system. Moreover, if $(A_n)_{n\in\mathbf{N}_+}$ is a denumerable collection of members of $\mathcal{M}$,

$$A_1 \cup ... \cup A_n = (A_1^c \cap ... \cap A_n^c)^c \in \mathcal{M}$$

for each $n$, since $\mathcal{M}$ is a $\lambda$-system and a $\pi$-system. Let $(B_n)_{n=1}^{\infty}$ be the disjungation of $(A_n)_{n=1}^{\infty}$. Then $(B_n)_{n\in\mathbf{N}_+}$ is a disjoint denumerable collection of members of $\mathcal{M}$ and Definition 1.2.2(c) implies that $\cup_{n=1}^{\infty}A_n = \cup_{n=1}^{\infty}B_n \in \mathcal{M}$.

**Theorem 1.2.2.** *Let $\mathcal{G}$ be a $\pi$-system and $\mathcal{D}$ a $\lambda$-system such that $\mathcal{G} \subseteq \mathcal{D}$. Then $\sigma(\mathcal{G}) \subseteq \mathcal{D}$.*

PROOF. Let $\mathcal{M}$ be the intersection of all $\lambda$-systems containing $\mathcal{G}$. The class $\mathcal{M}$ is a $\lambda$-system and $\mathcal{G} \subseteq \mathcal{M} \subseteq \mathcal{D}$. In view of Theorem 1.2.1 $\mathcal{M}$ is a $\sigma$-algebra, if $\mathcal{M}$ is a $\pi$-system and in that case $\sigma(\mathcal{G}) \subseteq \mathcal{M}$. Thus the theorem follows if we show that $\mathcal{M}$ is a $\pi$-system.

Given $C \subseteq X$, denote by $\mathcal{D}_C$ be the class of all $D \subseteq X$ such that $D \cap C \in \mathcal{M}$.

CLAIM 1. If $C \in \mathcal{M}$, then $\mathcal{D}_C$ is a $\lambda$-system.

PROOF OF CLAIM 1. First $X \in \mathcal{D}_C$ since $X \cap C = C \in \mathcal{M}$. Moreover, if $A, B \in \mathcal{D}_C$ and $A \subseteq B$, then $A \cap C, B \cap C \in \mathcal{M}$ and

$$(B \setminus A) \cap C = (B \cap C) \setminus (A \cap C) \in \mathcal{M}.$$

Accordingly from this, $B \setminus A \in \mathcal{D}_C$. Finally, if $(A_n)_{n\in\mathbf{N}_+}$ is a disjoint denumerable collection of members of $\mathcal{D}_C$, then $(A_n \cap C)_{n\in\mathbf{N}_+}$ is disjoint denumerable collection of members of $\mathcal{M}$ and

$$(\cup_{n\in\mathbf{N}_+}A_n) \cap C = \cup_{n\in\mathbf{N}_+}(A_n \cap C) \in \mathcal{M}.$$

Thus $\cup_{n\in\mathbf{N}_+} A_n \in \mathcal{D}_C$.

CLAIM 2. If $A \in \mathcal{G}$, then $\mathcal{M} \subseteq \mathcal{D}_A$.

PROOF OF CLAIM 2. If $B \in \mathcal{G}$, $A \cap B \in \mathcal{G} \subseteq \mathcal{M}$. Thus $B \in \mathcal{D}_A$. We have proved that $\mathcal{G} \subseteq \mathcal{D}_A$ and remembering that $\mathcal{M}$ is the intersection of all $\lambda$-systems containing $\mathcal{G}$ Claim 2 follows since $\mathcal{D}_A$ is a $\lambda$-system.

To complete the proof of Theorem 1.2.2, observe that $B \in \mathcal{D}_A$ if and only if $A \in \mathcal{D}_B$. By Claim 2, if $A \in \mathcal{G}$ and $B \in \mathcal{M}$, then $B \in \mathcal{D}_A$ that is $A \in \mathcal{D}_B$. Thus $\mathcal{G} \subseteq \mathcal{D}_B$ if $B \in \mathcal{M}$. Now the definition of $\mathcal{M}$ implies that $\mathcal{M} \subseteq \mathcal{D}_B$ if $B \in \mathcal{M}$. The proof is almost finished. In fact, if $A, B \in \mathcal{M}$ then $A \in \mathcal{D}_B$ that is $A \cap B \in \mathcal{M}$. Theorem 1.2.2 now follows from Theorem 1.2.1.

**Theorem 1.2.3.** *Let $\mu$ and $\nu$ be positive measures on $\mathcal{M} = \sigma(\mathcal{G})$, where $\mathcal{G}$ is a $\pi$-system, and suppose $\mu(A) = \nu(A)$ for every $A \in \mathcal{G}$.*

(a) *If $\mu$ and $\nu$ are probability measures, then $\mu = \nu$.*

(b) *Suppose there exist $E_n \in \mathcal{G}$, $n \in \mathbf{N}_+$, such that $X = \cup_{n=1}^{\infty} E_n$, $E_1 \subseteq E_2 \subseteq ...$, and*

$$\mu(E_n) = \nu(E_n) < \infty, \text{ all } n \in \mathbf{N}_+.$$

*Then $\mu = \nu$.*

PROOF. (a) Let

$$\mathcal{D} = \{A \in \mathcal{M};\ \ \mu(A) = \ \nu(A)\}.$$

It is immediate that $\mathcal{D}$ is a $\lambda$-system and Theorem 1.2.2 implies that $\mathcal{M} = \sigma(\mathcal{G}) \subseteq \mathcal{D}$ since $\mathcal{G} \subseteq \mathcal{D}$ and $\mathcal{G}$ is a $\pi$-system.

(b) If $\mu(E_n) = \nu(E_n) = 0$ for all all $n \in \mathbf{N}_+$, then

$$\mu(X) = \lim_{n\to\infty} \mu(E_n) = 0$$

and, in a similar way, $\nu(X) = 0$. Thus $\mu = \nu$. If $\mu(E_n) = \nu(E_n) > 0$, set

$$\mu_n(A) = \frac{1}{\mu(E_n)}\mu(A \cap E_n) \text{ and } \nu_n(A) = \frac{1}{\nu(E_n)}\nu(A \cap E_n)$$

for each $A \in \mathcal{M}$. By Part (a) $\mu_n = \nu_n$ and we get

$$\mu(A \cap E_n) = \nu(A \cap E_n)$$

for each $A \in \mathcal{M}$. Theorem 1.1.2(e) now proves that $\mu = \nu$.

Theorem 1.2.3 implies that there is at most one positive measure defined on $\mathcal{R}_n$ such that the measure of any open $n$-cell in $\mathbf{R}^n$ equals its volume.

Next suppose $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq Y$. The image of $A$ and the inverse image of $B$ are

$$f(A) = \{y;\ y = f(x) \text{ for some } x \in A\}$$

and

$$f^{-1}(B) = \{x;\ f(x) \in B\}$$

respectively. Note that

$$f^{-1}(Y) = X$$

and

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

Moreover, if $(A_i)_{i\in I}$ is a collection of subsets of $X$ and $(B_i)_{i\in I}$ is a collection of subsets of $Y$

$$f(\cup_{i\in I} A_i) = \cup_{i\in I} f(A_i)$$

and

$$f^{-1}(\cup_{i\in I} B_i) = \cup_{i\in I} f^{-1}(B_i).$$

Given a class $\mathcal{E}$ of subsets of $Y$, set

$$f^{-1}(\mathcal{E}) = \left\{f^{-1}(B);\ B \in \mathcal{E}\right\}.$$

If $(Y, \mathcal{N})$ is a measurable space, it follows that the class $f^{-1}(\mathcal{N})$ is a $\sigma$-algebra in $X$. If $(X, \mathcal{M})$ is a measurable space

$$\left\{B;\ f^{-1}(B) \in \mathcal{M}\right\}$$

is a $\sigma$-algebra in $Y$. Thus, given a class $\mathcal{E}$ of subsets of $Y$,

$$\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E})).$$

**Definition 1.2.3.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. The function $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(\mathcal{N}) \subseteq \mathcal{M}$. If we say that $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is measurable this means that $f : X \to Y$ is an $(\mathcal{M}, \mathcal{N})$-measurable function.

**Theorem 1.2.4.** *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and suppose $\mathcal{E}$ generates $\mathcal{N}$. The function $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if*

$$f^{-1}(\mathcal{E}) \subseteq \mathcal{M}.$$

PROOF. The assumptions yield

$$\sigma(f^{-1}(\mathcal{E})) \subseteq \mathcal{M}.$$

Since

$$\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E})) = f^{-1}(\mathcal{N})$$

we are done.

**Corollary 1.2.1.** *A function $f : X \to \mathbf{R}$ is $(\mathcal{M}, \mathcal{R})$-measurable if and only if the set $f^{-1}(]\alpha, \infty[) \in \mathcal{M}$ for all $\alpha \in \mathbf{R}$.*

If $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and $\mu$ is a positive measure on $\mathcal{M}$, the equation

$$\nu(B) = \mu(f^{-1}(B)),\ B \in \mathcal{N}$$

defines a positive measure $\nu$ on $\mathcal{N}$. We will write $\nu = \mu f^{-1}$, $\nu = f(\mu)$ or $\nu = \mu_f$. The measure $\nu$ is called the image measure of $\mu$ under $f$ and $f$ is said to transport $\mu$ to $\nu$. Two $(\mathcal{M}, \mathcal{N})$-measurable functions $f : X \to Y$ and $g : X \to Y$ are said to be $\mu$-equimeasurable if $f(\mu) = g(\mu)$.

As an example, let $a \in \mathbf{R}^n$ and define $f(x) = x + a$ if $x \in \mathbf{R}^n$. If $B \subseteq \mathbf{R}^n$,

$$f^{-1}(B) = \{x;\ x + a \in B\} = B - a.$$

Thus $f^{-1}(B)$ is an open $n$-cell if $B$ is, and Theorem 1.2.4 proves that $f$ is $(\mathcal{R}_n, \mathcal{R}_n)$-measurable. Now, granted the existence of volume measure $v_n$, for every $B \in \mathcal{R}_n$ define

$$\mu(B) = f(v_n)(B) = v_n(B - a).$$

Then $\mu(B) = v_n(B)$ if $B$ is an open $n$-cell and Theorem 1.2.3 implies that $\mu = v_n$. We have thus proved the following

**Theorem 1.2.5.** *For any $A \in \mathcal{R}_n$ and $x \in \mathbf{R}^n$*

$$A + x \in \mathcal{R}_n$$

*and*

$$v_n(A + x) = v_n(A).$$

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. A measurable function $\xi$ defined on $\Omega$ is called a random variable and the image measure $P_\xi$ is called the probability law of $\xi$. We sometimes write

$$\mathcal{L}(\xi) = P_\xi.$$

Here are two simple examples.

If the range of a random variable $\xi$ consists of $n$ points $S = \{s_1, ..., s_n\}$ $(n \geq 1)$ and $P_\xi = \frac{1}{n} c_S$, $\xi$ is said to have a uniform distribution in $S$. Note that

$$P_\xi = \frac{1}{n} \Sigma_{k=1}^n \delta_{s_k}.$$

Suppose $\lambda > 0$ is a constant. If a random variable $\xi$ has its range in $\mathbf{N}$ and

$$P_\xi = \Sigma_{n=0}^\infty \frac{\lambda^n}{n!} e^{-\lambda} \delta_n$$

then $\xi$ is said to have a Poisson distribution with parameter $\lambda$.

**Exercises**

1. Let $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$. Show that

$$f(f^{-1}(B)) \subseteq B \text{ and } f^{-1}(f(A)) \supseteq A.$$

2. Let $(X, \mathcal{M})$ be a measurable space and suppose $A \subseteq X$. Show that the function $\chi_A$ is $(\mathcal{M}, \mathcal{R})$-measurable if and only if $A \in \mathcal{M}$.

3. Suppose $(X, \mathcal{M})$ is a measurable space and $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}$, a sequence of $(\mathcal{M}, \mathcal{R})$-measurable functions such that

$$\lim_{n\to\infty} f_n(x) \text{ exists and } = f(x) \in \mathbf{R}$$

for each $x \in X$. Prove that $f$ is $(\mathcal{M}, \mathcal{R})$-measurable.

4. Suppose $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ and $g : (Y, \mathcal{N}) \to (Z, \mathcal{S})$ are measurable. Prove that $g \circ f$ is $(\mathcal{M}, \mathcal{S})$-measurable.

5. Granted the existence of volume measure $v_n$, show that $v_n(rA) = r^n v_n(A)$ if $r \geq 0$ and $A \in \mathcal{R}$.

6. Let $\mu$ be the counting measure on $\mathbf{Z}^2$ and $f(x, y) = x$, $(x, y) \in \mathbf{Z}^2$. The measure $\mu$ is $\sigma$-finite. Prove that the image measure $f(\mu)$ is not $\sigma$-finite.

7. Let $\mu, \nu : \mathcal{R} \to [0, \infty]$ be two positive measures such that $\mu(I) = \nu(I) < \infty$ for each open subinterval of $\mathbf{R}$. Prove that $\mu = \nu$.

8. Suppose $\xi$ has a Poisson distribution with parameter $\lambda$. Show that $P_\xi\left[2\mathbf{N}\right] = e^{-\lambda}\cosh\lambda$.

9. Find a $\lambda$-system which is not a $\sigma$-algebra.

### 1.3. Lebesgue Measure

Once the problem about the existence of volume measure is solved the existence of the so called Lebesgue measure is simple to establish as will be seen in this section. We start with some concepts of general interest.

If $(X, \mathcal{M}, \mu)$ is a positive measure space, the zero set $\mathcal{Z}_\mu$ of $\mu$ is, by definition, the set at all $A \in \mathcal{M}$ such that $\mu(A) = 0$. An element of $\mathcal{Z}_\mu$ is called a null set or $\mu$-null set. If

$$(A \in \mathcal{Z}_\mu \text{ and } B \subseteq A) \Rightarrow B \in \mathcal{M}$$

the measure space $(X, \mathcal{M}, \mu)$ is said to be complete. In this case the measure $\mu$ is also said to be complete. The positive measure space $(X, \{\phi, X\}, \mu)$, where $X = \{0, 1\}$ and $\mu = 0$, is not complete since $X \in \mathcal{Z}_\mu$ and $\{0\} \notin \{\phi, X\}$.

**Theorem 1.3.1** *If* $(E_n)_{n=1}^\infty$ *is a denumerable collection of members of* $\mathcal{Z}_\mu$ *then* $\cup_{n=1}^\infty E_n \in \mathcal{Z}_\mu$.

PROOF We have

$$0 \leq \mu(\cup_{n=1}^\infty E_n) \leq \Sigma_{n=1}^\infty \mu(E_n) = 0$$

which proves the result.

Granted the existence of linear measure $v_1$ it follows from Theorem 1.3.1 that $\mathbf{Q} \in \mathcal{Z}_{v_1}$ since $\mathbf{Q}$ is countable and $\{a\} \in \mathcal{Z}_{v_1}$ for each real number $a$.

Suppose $(X, \mathcal{M}, \mu)$ is an arbitrary positive measure space. It turns out that $\mu$ is the restriction to $\mathcal{M}$ of a complete measure. To see this suppose $\mathcal{M}^-$ is the class of all $E \subseteq X$ is such that there exist sets $A, B \in \mathcal{M}$ such that

$A \subseteq E \subseteq B$ and $B \setminus A \in \mathcal{Z}_\mu$. It is obvious that $X \in \mathcal{M}^-$ since $\mathcal{M} \subseteq \mathcal{M}^-$. If $E \in \mathcal{M}^-$, choose $A, B \in \mathcal{M}$ such that $A \subseteq E \subseteq B$ and $B \setminus A \in \mathcal{Z}_\mu$. Then $B^c \subseteq E^c \subseteq A^c$ and $A^c \setminus B^c = B \setminus A \in \mathcal{Z}_\mu$ and we conclude that $E^c \in \mathcal{M}^-$. If $(E_i)_{i=1}^\infty$ is a denumerable collection of members of $\mathcal{M}^-$, for each $i$ there exist sets $A_i, B_i \in \mathcal{M}$ such that $A_i \subseteq E \subseteq B_i$ and $B_i \setminus A_i \in \mathcal{Z}_\mu$. But then

$$\cup_{i=1}^\infty A_i \subseteq \cup_{i=1}^\infty E_i \subseteq \cup_{i=1}^\infty B_i$$

where $\cup_{i=1}^\infty A_i, \cup_{i=1}^\infty B_i \in \mathcal{M}$. Moreover, $(\cup_{i=1}^\infty B_i) \setminus (\cup_{i=1}^\infty A_i) \in \mathcal{Z}_\mu$ since

$$(\cup_{i=1}^\infty B_i) \setminus (\cup_{i=1}^\infty A_i) \subseteq \cup_{i=1}^\infty (B_i \setminus A_i).$$

Thus $\cup_{i=1}^\infty E_i \in \mathcal{M}^-$ and $\mathcal{M}^-$ is a $\sigma$-algebra.

If $E \in \mathcal{M}$, suppose $A_i, B_i \in \mathcal{M}$ are such that $A_i \subseteq E \subseteq B_i$ and $B_i \setminus A_i \in \mathcal{Z}_\mu$ for $i = 1, 2$. Then for each $i$, $(B_1 \cap B_2) \setminus A_i \in \mathcal{Z}_\mu$ and

$$\mu(B_1 \cap B_2) = \mu((B_1 \cap B_2) \setminus A_i) + \mu(A_i) = \mu(A_i).$$

Thus the real numbers $\mu(A_1)$ and $\mu(A_2)$ are the same and we define $\bar{\mu}(E)$ to be equal to this common number. Note also that $\mu(B_1) = \bar{\mu}(E)$. It is plain that $\bar{\mu}(\phi) = 0$. If $(E_i)_{i=1}^\infty$ is a disjoint denumerable collection of members of $\mathcal{M}$, for each $i$ there exist sets $A_i, B_i \in \mathcal{M}$ such that $A_i \subseteq E_i \subseteq B_i$ and $B_i \setminus A_i \in \mathcal{Z}_\mu$. From the above it follows that

$$\bar{\mu}(\cup_{i=1}^\infty E_i) = \mu(\cup_{i=1}^\infty A_i) = \Sigma_{n=1}^\infty \mu(A_i) = \Sigma_{n=1}^\infty \bar{\mu}(E_i).$$

We have proved that $\bar{\mu}$ is a positive measure on $\mathcal{M}^-$. If $E \in \mathcal{Z}_{\bar{\mu}}$ the definition of $\bar{\mu}$ shows that any set $A \subseteq E$ belongs to the $\sigma$-algebra $\mathcal{M}^-$. It follows that the measure $\bar{\mu}$ is complete and its restriction to $\mathcal{M}$ equals $\mu$.

The measure $\bar{\mu}$ is called the completion of $\mu$ and $\mathcal{M}^-$ is called the completion of $\mathcal{M}$ with respect to $\mu$.

**Definition 1.3.1** The completion of volume measure $v_n$ on $\mathbf{R}^n$ is called Lebesgue measure on $\mathbf{R}^n$ and is denoted by $m_n$. The completion of $\mathcal{R}_n$ with respect to $v_n$ is called the Lebesgue $\sigma$-algebra in $\mathbf{R}^n$ and is denoted by $\mathcal{R}_n^-$. A member of the class $\mathcal{R}_n^-$ is called a Lebesgue measurable set in $\mathbf{R}^n$ or a Lebesgue set in $\mathbf{R}^n$. A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be Lebesgue measurable if it is $(\mathcal{R}_n^-, \mathcal{R})$-measurable. Below, $m_1$ is written $m$ if this notation will not lead to misunderstanding. Furthermore, $\mathcal{R}_1^-$ is written $\mathcal{R}^-$.

**Theorem 1.3.2.** *Suppose* $E \in \mathcal{R}_n^-$ *and* $x \in \mathbf{R}^n$. *Then* $E + x \in \mathcal{R}_n^-$ *and* $m_n(E+x) = m_n(E)$.

PROOF. Choose $A, B \in \mathcal{R}_n$ such that $A \subseteq E \subseteq B$ and $B \setminus A \in \mathcal{Z}_{v_n}$. Then, by Theorem 1.2.5, $A + x, B + x \in \mathcal{R}_n$, $v_n(A+x) = v_n(A) = m_n(E)$, and $(A+x) \setminus (B+x) = (A \setminus B) + x \in \mathcal{Z}_{v_n}$. Since $A + x \subseteq E + x \subseteq B + x$ the theorem is proved.

The Lebesgue $\sigma$-algebra in $\mathbf{R}^n$ is very large and contains each set of interest in analysis and probability. In fact, in most cases, the $\sigma$-algebra $\mathcal{R}_n$ is sufficiently large but there are some exceptions. For example, if $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $A \in \mathcal{R}_n$, the image set $f(A)$ need not belong to the class $\mathcal{R}_n$ (see e.g. the Dudley book [D]). To prove the existence of a subset of the real line, which is not Lebesgue measurable we will use the so called Axiom of Choice.

**Axiom of Choice.** If $(A_i)_{i \in I}$ is a non-empty collection of non-empty sets, there exists a function $f : I \to \cup_{i \in I} A_i$ such that $f(i) \in A_i$ for every $i \in I$.

Let $X$ and $Y$ be sets. The set of all ordered pairs $(x, y)$, where $x \in X$ and $y \in Y$ is denoted by $X \times Y$. An arbitrary subset $R$ of $X \times Y$ is called a relation. If $(x, y) \in R$, we write $x \sim y$. A relation is said to be an equivalence relation on $X$ if $X = Y$ and

(i) $x \sim x$ (reflexivity)
(ii) $x \sim y \Rightarrow y \sim x$ (symmetry)
(ii) $(x \sim y$ and $y \sim z) \Rightarrow x \sim z$ (transitivity)

The equivalence class $R(x) =_{def} \{y;\ y \sim x\}$. The definition of the equivalence relation $\sim$ implies the following:

(a) $x \in R(x)$

(b) $R(x) \cap R(y) \neq \phi \Rightarrow R(x) = R(y)$
(c) $\cup_{x \in X} R(x) = X$.

An equivalence relation leads to a partition of $X$ into a disjoint collection of subsets of $X$.

Let $X = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and define an equivalence relation for numbers $x, y$ in $X$ by stating that $x \sim y$ if $x - y$ is a rational number. By the Axiom of Choice it is possible to pick exactly one element from each equivalence class. Thus there exists a subset $NL$ of $X$ which contains exactly one element from each equivalence class.

If we assume that $NL \in \mathcal{R}^-$ we get a contradiction as follows. Let $(r_i)_{i=1}^{\infty}$ be an enumeration of the rational numbers in $[-1, 1]$. Then

$$X \subseteq \cup_{i=1}^{\infty} (r_i + NL)$$

and it follows from Theorem 1.3.1 that $r_i + NL \notin \mathcal{Z}_m$ for some $i$. Thus, by Theorem 1.3.2, $NL \notin \mathcal{Z}_m$.

Now assume $(r_i + NL) \cap (r_j + NL) \neq \phi$. Then there exist $a', a'' \in NL$ such that $r_i + a' = r_j + a''$ or $a' - a'' = r_j - r_i$. Hence $a' \sim a''$ and it follows that $a'$ and $a''$ belong to the same equivalence class. But then $a' = a''$. Thus $r_i = r_j$ and we conclude that $(r_i + NL)_{i \in \mathbf{N}_+}$ is a disjoint enumeration of Lebesgue sets. Now, since

$$\cup_{i=1}^{\infty} (r_i + NL) \subseteq \left[-\frac{3}{2}, \frac{3}{2}\right]$$

it follows that

$$3 \geq m(\cup_{i=1}^{\infty} (r_i + NL)) = \Sigma_{n=1}^{\infty} m(NL).$$

But then $NL \in \mathcal{Z}_m$, which is a contradiction. Thus $NL \notin \mathcal{R}^-$.

In the early 1970' Solovay [$S$] proved that it is consistent with the usual axioms of Set Theory, excluding the Axiom of Choice, that every subset of $\mathbf{R}$ is Lebesgue measurable.

From the above we conclude that the Axiom of Choice implies the existence of a subset of the set of real numbers which does not belong to the class $\mathcal{R}$. Interestingly enough, such an example can be given without any use of the Axiom of Choice and follows naturally from the theory of analytic sets. The interested reader may consult the Dudley book [$D$] .

**Exercises**

1. $(X, \mathcal{M}, \mu)$ is a positive measure space. Prove or disprove: If $A \subseteq E \subseteq B$ and $\mu(A) = \mu(B)$ then $E$ belongs to the domain of the completion $\bar{\mu}$.

2. Prove or disprove: If $A$ and $B$ are not Lebesgue measurable subsets of $\mathbf{R}$, then $A \cup B$ is not Lebesgue measurable.

3. Let $(X, \mathcal{M}, \mu)$ be a complete positive measure space and suppose $A, B \in \mathcal{M}$, where $B \setminus A$ is a $\mu$-null set. Prove that $E \in \mathcal{M}$ if $A \subseteq E \subseteq B$ (stated otherwise $\mathcal{M}^- = \mathcal{M}$).

4. Suppose $E \subseteq \mathbf{R}$ and $E \notin \mathcal{R}^-$. Show there is an $\varepsilon > 0$ such that

$$m(B \setminus A) \geq \varepsilon$$

for all $A, B \in \mathcal{R}^-$ such that $A \subseteq E \subseteq B$.

5. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $(Y, \mathcal{N})$ a measurable space. Furthermore, suppose $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and let $\nu = \mu f^{-1}$, that is $\nu(B) = \mu(f^{-1}(B))$, $B \in \mathcal{N}$. Show that $f$ is $(\mathcal{M}^-, \mathcal{N}^-)$-measurable, where $\mathcal{M}^-$ denotes the completion of $\mathcal{M}$ with respect to $\mu$ and $\mathcal{N}^-$ the completion of $\mathcal{N}$ with respect to $\nu$.

**1.4. Carathéodory's Theorem**

In these notes we exhibit two famous approaches to Lebesgue measure. One is based on the Carathéodory Theorem, which we present in this section, and the other one, due to F. Riesz, is a representation theorem of positive linear functionals on spaces of continuous functions in terms of positive measures. The latter approach, is presented in Chapter 3. Both methods depend on topological concepts such as compactness.

**Definition 1.4.1.** A function $\theta : \mathcal{P}(X) \to [0, \infty]$ is said to be an outer measure if the following properties are satisfied:

(i) $\theta(\phi) = 0$.
(ii) $\theta(A) \leq \theta(B)$ if $A \subseteq B$.
(iii) for any denumerable collection $(A_n)_{n=1}^{\infty}$ of subsets of $X$

$$\theta(\cup_{n=1}^{\infty} A_n) \leq \Sigma_{n=1}^{\infty} \theta(A_n).$$

Since

$$E = (E \cap A) \cup (E \cap A^c)$$

an outer measure $\theta$ satisfies the inequality

$$\theta(E) \leq \theta(E \cap A) + \theta(E \cap A^c).$$

If $\theta$ is an outer measure on $X$ we define $\mathcal{M}(\theta)$ as the set of all $A \subseteq X$ such that

$$\theta(E) = \theta(E \cap A) + \theta(E \cap A^c) \text{ for all } E \subseteq X$$

or, what amounts to the same thing,

$$\theta(E) \geq \theta(E \cap A) + \theta(E \cap A^c) \text{ for all } E \subseteq X.$$

The next theorem is one of the most important in measure theory.

**Theorem 1.4.1. ( Carathéodory's Theorem)** *Suppose $\theta$ is an outer measure. The class $\mathcal{M}(\theta)$ is a $\sigma$-algebra and the restriction of $\theta$ to $\mathcal{M}(\theta)$ is a complete measure.*

PROOF. Clearly, $\phi \in \mathcal{M}(\theta)$ and $A^c \in \mathcal{M}(\theta)$ if $A \in \mathcal{M}(\theta)$. Moreover, if $A, B \in \mathcal{M}(\theta)$ and $E \subseteq X$,

$$\theta(E) = \theta(E \cap A) + \theta(E \cap A^c)$$

$$= \theta(E \cap A \cap B) + \theta(E \cap A \cap B^c)$$

$$+\theta(E \cap A^c \cap B) + \theta(E \cap A^c \cap B^c).$$

But

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

and

$$A^c \cap B^c = (A \cup B)^c$$

and we get

$$\theta(E) \geq \theta(E \cap (A \cup B)) + \theta(E \cap (A \cup B)^c).$$

It follows that $A \cup B \in \mathcal{M}(\theta)$ and we have proved that the class $\mathcal{M}(\theta)$ is an algebra. Now if $A, B \in \mathcal{M}(\theta)$ are disjoint

$$\theta(A \cup B) = \theta((A \cup B) \cap A) + \theta((A \cup B) \cap A^c) = \theta(A) + \theta(B)$$

and therefore the restriction of $\theta$ to $\mathcal{M}(\theta)$ is a content.

Next we prove that $\mathcal{M}(\theta)$ is a $\sigma$-algebra. Let $(A_i)_{i=1}^{\infty}$ be a disjoint denumerable collection of members of $\mathcal{M}(\theta)$ and set for each $n \in \mathbf{N}$

$$B_n = \cup_{1 \leq i \leq n} A_i \text{ and } B = \cup_{i=1}^{\infty} A_i$$

(here $B_0 = \phi$). Then for any $E \subseteq X$,

$$\theta(E \cap B_n) = \theta(E \cap B_n \cap A_n) + \theta(E \cap B_n \cap A_n^c)$$

$$= \theta(E \cap A_n) + \theta(E \cap B_{n-1})$$

and, by induction,

$$\theta(E \cap B_n) = \Sigma_{i=1}^{n} \theta(E \cap A_i).$$

But then

$$\theta(E) = \theta(E \cap B_n) + \theta(E \cap B_n^c)$$

$$\geq \Sigma_{i=1}^{n}\theta(E\cap A_i)+\theta(E\cap B^c)$$

and letting $n\to\infty$,

$$\theta(E)\geq \Sigma_{i=1}^{\infty}\theta(E\cap A_i)+\theta(E\cap B^c)$$

$$\geq \theta(\cup_{i=1}^{\infty}(E\cap A_i))+\theta(E\cap B^c)$$

$$=\theta(E\cap B)+\theta(E\cap B^c)\geq\theta(E).$$

All the inequalities in the last calculation must be equalities and we conclude that $B\in\mathcal{M}(\theta)$ and, choosing $E=B$, results in

$$\theta(B)=\Sigma_{i=1}^{\infty}\theta(A_i).$$

Thus $\mathcal{M}(\theta)$ is a $\sigma$-algebra and the restriction of $\theta$ to $\mathcal{M}(\theta)$ is a positive measure.

Finally we prove that the the restriction of $\theta$ to $\mathcal{M}(\theta)$ is a complete measure. Suppose $B\subseteq A\in\mathcal{M}(\theta)$ and $\theta(A)=0$. If $E\subseteq X$,

$$\theta(E)\leq\theta(E\cap B)+\theta(E\cap B^c)\leq\theta(E\cap B^c)\leq\theta(E)$$

and so $B\in\mathcal{M}(\theta)$. The theorem is proved.

**Exercises**

1. Suppose $\theta_i:\mathcal{P}(X)\to[0,\infty[\,,\ i=1,2$, are outer measures. Prove that $\theta=\max(\theta_1,\theta_2)$ is an outer measure.

2. Suppose $a,b\in\mathbf{R}$ and $a\neq b$. Set $\theta=\max(\delta_a,\delta_b)$. Prove that

$$\{a\}\,,\{b\}\notin\mathcal{M}(\theta).$$

## 1.5. Existence of Linear Measure

The purpose of this section is to show the existence of linear measure on $\mathbf{R}$ using the Carathéodory Theorem and a minimum of topology.

First let us recall the definition of infimum and supremum of a non-empty subset of the extended real line. Suppose $A$ is a non-empty subset of $[-\infty, \infty] = \mathbf{R} \cup \{-\infty, \infty\}$. We define $-\infty \leq x$ and $x \leq \infty$ for all $x \in [-\infty, \infty]$. An element $b \in [-\infty, \infty]$ is called a majorant of $A$ if $x \leq b$ for all $x \in A$ and a minorant if $x \geq b$ for all $x \in A$. The Supremum Axiom states that $A$ possesses a least majorant, which is denoted by $\sup A$. From this follows that if $A$ is non-empty, then $A$ possesses a greatest minorant, which is denoted by $\inf A$. (Actually, the Supremum Axiom is a theorem in courses where time is spent on the definition of real numbers.)

**Theorem 1.5.1. (The Heine-Borel Theorem; weak form)** *Let $[a, b]$ be a closed bounded interval and $(U_i)_{i \in I}$ a collection of open sets such that*

$$\cup_{i \in I} U_i \supseteq [a, b] \, .$$

*Then*

$$\cup_{i \in J} U_i \supseteq [a, b]$$

*for some finite subset $J$ of $I$.*

PROOF. Let $A$ be the set of all $x \in [a, b]$ such that

$$\cup_{i \in J} U_i \supseteq [a, x]$$

for some finite subset $J$ of $I$. Clearly, $a \in A$ since $a \in U_i$ for some $i$. Let $c = \sup A$. There exists an $i_0$ such that $c \in U_{i_0}$. Let $c \in \, ]a_0, b_0[ \, \subseteq U_{i_0}$, where $a_0 < b_0$. Furthermore, by the very definition of least upper bound, there exists a finite set $J$ such that

$$\cup_{i \in J} U_i \supseteq [a, (a_0 + c)/2] \, .$$

Hence

$$\cup_{i \in J \cup \{i_0\}} U_k \supseteq [a, (c + b_0)/2]$$

and it follows that $c \in A$ and $c = b$. The lemma is proved.

A subset $K$ of $\mathbf{R}$ is called compact if for every family of open subsets $U_i$, $i \in I$, with $\cup_{i\in I} U_i \supseteq K$ we have $\cup_{i\in J} U_i \supseteq K$ for some finite subset $J$ of $I$. The Heine-Borel Theorem shows that a closed bounded interval is compact.

If $x, y \in \mathbf{R}$ and $E$, $F \subseteq \mathbf{R}$, let

$$d(x, y) =\mid x - y \mid$$

be the distance between $x$ and $y$, let

$$d(x, E) = \inf_{u\in E} d(x, u)$$

be the distance from $x$ to $E$, and let

$$d(E, F) = \inf_{u\in E, v\in F} d(u, v)$$

be the distance between $E$ and $F$ (here the infimum of the emty set equals $\infty$). Note that for any $u \in E$,

$$d(x, u) \leq d(x, y) + d(y, u)$$

and, hence

$$d(x, E) \leq d(x, y) + d(y, u)$$

and

$$d(x, E) \leq d(x, y) + d(y, E).$$

By interchanging the roles of $x$ and $y$ and assuming that $E \neq \phi$, we get

$$\mid d(x, E) - d(y, E) \mid \leq d(x, y).$$

Note that if $F \subseteq \mathbf{R}$ is closed and $x \notin F$, then $d(x, F) > 0$.

An outer measure $\theta : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ is called a metric outer measure if

$$\theta(A \cup B) = \theta(A) + \theta(B)$$

for all $A, B \in \mathcal{P}(\mathbf{R})$ such that $d(A, B) > 0$.

**Theorem 1.5.2.** *If* $\theta : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ *is a metric outer measure, then* $\mathcal{R} \subseteq \mathcal{M}(\theta)$.

PROOF. Let $F \in \mathcal{P}(\mathbf{R})$ be closed. It is enough to show that $F \in \mathcal{M}(\theta)$. To this end we choose $E \subseteq X$ with $\theta(E) < \infty$ and prove that

$$\theta(E) \geq \theta(E \cap F) + \theta(E \cap F^c).$$

Let $n \geq 1$ be an integer and define

$$A_n = \left\{ x \in E \cap F^c;\ d(x, F) \geq \frac{1}{n} \right\}.$$

Note that $A_n \subseteq A_{n+1}$ and

$$E \cap F^c = \cup_{n=1}^{\infty} A_n.$$

Moreover, since $\theta$ is a metric outer measure

$$\theta(E) \geq \theta((E \cap F) \cup A_n) = \theta(E \cap F) + \theta(A_n)$$

and, hence, proving

$$\theta(E \cap F^c) = \lim_{n \to \infty} \theta(A_n)$$

we are done.

Let $B_n = A_{n+1} \cap A_n^c$. It is readily seen that

$$d(B_{n+1}, A_n) \geq \frac{1}{n(n+1)}$$

since if $x \in B_{n+1}$ and

$$d(x, y) < \frac{1}{n(n+1)}$$

then

$$d(y, F) \leq d(y, x) + d(x, F) < \frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}.$$

Now

$$\theta(A_{2k+1}) \geq \theta(B_{2k} \cup A_{2k-1}) = \theta(B_{2k}) + \theta(A_{2k-1})$$

$$\geq ... \geq \Sigma_{i=1}^{k} \theta(B_{2i})$$

and in a similar way

$$\theta(A_{2k}) \geq \Sigma_{i=1}^{k} \theta(B_{2i-1}).$$

But $\theta(A_n) \leq \theta(E) < \infty$ and we conclude that

$$\Sigma_{i=1}^{\infty}\theta(B_i) < \infty.$$

We now use that

$$E \cap F^c = A_n \cup (\cup_{i=n}^{\infty} B_i)$$

to obtain

$$\theta(E \cap F^c) \leq \theta(A_n) + \Sigma_{i=n}^{\infty}\theta(B_i).$$

Now, since $\theta(E \cap F^c) \geq \theta(A_n)$,

$$\theta(E \cap F^c) = \lim_{n\to\infty} \theta(A_n)$$

and the theorem is proved.

PROOF OF THEOREM 1.1.1 IN ONE DIMENSION. Suppose $\delta > 0$. If $A \subseteq \mathbf{R}$, define

$$\theta_\delta(A) = \inf \Sigma_{k=1}^{\infty} l(I_k)$$

the infimum being taken over all open intervals $I_k$ with $l(I_k) < \delta$ such that

$$A \subseteq \cup_{k=1}^{\infty} I_k.$$

Obviously, $\theta_\delta(\phi) = 0$ and $\theta_\delta(A) \leq \theta_\delta(B)$ if $A \subseteq B$. Suppose $(A_n)_{n=1}^{\infty}$ is a denumerable collection of subsets of $\mathbf{R}$ and let $\varepsilon > 0$. For each $n$ there exist intervals $I_{kn}, k \in \mathbf{N}_+$, such that $l(I_{kn}) < \delta$,

$$A_n \subseteq \cup_{k=1}^{\infty} I_{kn}$$

and

$$\Sigma_{k=1}^{\infty} l(I_{kn}) \leq \theta_\delta(A_n) + \varepsilon 2^{-n}.$$

Then

$$A =_{def} \cup_{n=1}^{\infty} A_n \subseteq \cup_{k,n=1}^{\infty} I_{kn}$$

and

$$\Sigma_{k,n=1}^{\infty} l(I_{kn}) \leq \Sigma_{n=1}^{\infty}\theta_\delta(A_n) + \varepsilon.$$

Thus

$$\theta_\delta(A) \leq \Sigma_{n=1}^{\infty}\theta_\delta(A_n) + \varepsilon$$

and, since $\varepsilon > 0$ is arbitrary,

$$\theta_\delta(A) \leq \Sigma_{n=1}^{\infty} \theta_\delta(A_n).$$

It follows that $\theta_\delta$ is an outer measure.

If $I$ is an open interval it is simple to see that

$$\theta_\delta(I) \leq l(I).$$

To prove the reverse inequality, choose a closed bounded interval $J \subseteq I$. Now, if

$$I \subseteq \cup_{k=1}^{\infty} I_k$$

where each $I_k$ is an open interval of $l(I_k) < \delta$, it follows from the Heine-Borel Theorem that

$$J \subseteq \cup_{k=1}^{n} I_k$$

for some $n$. Hence

$$l(J) \leq \Sigma_{k=1}^{n} l(I_k) \leq \Sigma_{k=1}^{\infty} l(I_k)$$

and it follows that

$$l(J) \leq \theta_\delta(I)$$

and, accordingly from this,

$$l(I) \leq \theta_\delta(I).$$

Thus, if $I$ is an open interval, then

$$\theta_\delta(I) = l(I).$$

Note that $\theta_{\delta_1} \geq \theta_{\delta_2}$ if $0 < \delta_1 \leq \delta_2$. We define

$$\theta_0(A) = \lim_{\delta \to 0} \theta_\delta(A) \text{ if } A \subseteq \mathbf{R}.$$

It obvious that $\theta_0$ is an outer measure such that $\theta_0(I) = l(I)$, if $I$ is an open interval.

To complete the proof we show that $\theta_0$ is a metric outer measure. To this end let $A, B \subseteq \mathbf{R}$ and $d(A, B) > 0$. Suppose $0 < \delta < d(A, B)$ and

$$A \cup B \subseteq \cup_{k=1}^{\infty} I_k$$

where each $I_k$ is an open interval with $l(I_k) < \delta$. Let

$$\alpha = \{k;\ I_k \cap A \neq \phi\}$$

and

$$\beta = \{k;\ I_k \cap B \neq \phi\}\,.$$

Then $\alpha \cap \beta = \phi$,

$$A \subseteq \cup_{k\in\alpha} I_k$$

and

$$B \subseteq \cup_{k\in\beta} I_k$$

and it follows that

$$\Sigma_{k=1}^{\infty} l(I_k) \geq \Sigma_{k\in\alpha} l(I_k) + \Sigma_{k\in\beta} l(I_k)$$

$$\geq \theta_\delta(A) + \theta_\delta(B).$$

Thus

$$\theta_\delta(A \cup B) \geq \theta_\delta(A) + \theta_\delta(B)$$

and by letting $\delta \to 0$ we have

$$\theta_0(A \cup B) \geq \theta_0(A) + \theta_0(B)$$

and

$$\theta_0(A \cup B) = \theta_0(A) + \theta_0(B).$$

Finally by applying the Carathéodory Theorem and Theorem 1.5.2 it follows that the restriction of $\theta_0$ to $\mathcal{R}$ equals $v_1$.

We end this section with some additional results of great interest.

**Theorem 1.5.3.** *For any* $\delta > 0$, $\theta_\delta = \theta_0$. *Moreover, if* $A \subseteq \mathbf{R}$

$$\theta_0(A) = \inf \Sigma_{k=1}^{\infty} l(I_k)$$

*the infimum being taken over all open intervals* $I_k$, $k \in \mathbf{N}_+$, *such that* $\cup_{k=1}^{\infty} I_k \supseteq A$.

PROOF. It follows from the definition of $\theta_0$ that $\theta_\delta \leq \theta_0$. To prove the reverse inequality let $A \subseteq \mathbf{R}$ and choose open intervals $I_k$, $k \in \mathbf{N}_+$, such that $\cup_{k=1}^{\infty} I_k \supseteq A$. Then

$$\theta_0(A) \leq \theta_0(\cup_{k=1}^{\infty} I_k) \leq \Sigma_{k=1}^{\infty} \theta_0(I_k)$$

$$= \Sigma_{k=1}^{\infty} l(I_k).$$

Hence

$$\theta_0(A) \leq \inf \Sigma_{k=1}^{\infty} l(I_k)$$

the infimum being taken over all open intervals $I_k$, $k \in \mathbf{N}_+$, such that $\cup_{k=1}^{\infty} I_k \supseteq A$. Thus $\theta_0(A) \leq \theta_\delta(A)$, which completes the proof of Theorem 1.5.3.

**Theorem 1.5.4.** *If* $A \subseteq \mathbf{R}$,

$$\theta_0(A) = \inf_{\substack{U \supseteq A \\ U \text{ open}}} \theta_0(U).$$

*Moreover, if* $A \in \mathcal{M}(\theta_0)$,

$$\theta_0(A) = \sup_{\substack{K \subseteq A \\ K \text{ closed bounded interval}}} \theta_0(K).$$

PROOF. If $A \subseteq U$, $\theta_0(A) \leq \theta_0(U)$. Hence

$$\theta_0(A) \leq \inf_{\substack{U \supseteq A \\ U \text{ open}}} \theta_0(U).$$

Next let $\varepsilon > 0$ be fixed and choose open intervals $I_k$, $k \in \mathbf{N}_+$, such that $\cup_{k=1}^{\infty} I_k \supseteq A$ and

$$\Sigma_{k=1}^{\infty} l(I_k) \leq \theta_0(A) + \varepsilon$$

(here observe that it may happen that $\theta_0(A) = \infty$). Then the set $U =_{def} \cup_{k=1}^{\infty} I_k$ is open and

$$\theta_0(U) \leq \Sigma_{k=1}^{\infty} \theta_0(I_k) \leq \Sigma_{k=1}^{\infty} l(I_k) \leq \theta_0(A) + \varepsilon.$$

Thus

$$\inf_{\substack{U \supseteq A \\ U \text{ open}}} \theta_0(U) \leq \theta_0(A)$$

and we have proved that

$$\theta_0(A) = \inf_{\substack{U \supseteq A \\ U \text{ open}}} \theta_0(U).$$

If $K \subseteq A$, $\theta_0(K) \leq \theta_0(A)$ and, accordingly from this,

$$\sup_{\substack{K \subseteq A \\ K \text{ closed bounded}}} \theta_0(K) \leq \theta_0(A).$$

To prove the reverse inequality we first assume that $A \in \mathcal{M}(\theta_0)$ is bounded. Let $\varepsilon > 0$ be fixed and suppose $J$ is a closed bounded interval containing $A$. Then we know from the first part of Theorem 1.5.4 already proved that there exists an open set $U \supseteq J \setminus A$ such that

$$\theta_0(U) < \theta_0(J \setminus A) + \varepsilon.$$

But then

$$\theta_0(J) \leq \theta_0(J \setminus U) + \theta_0(U) < \theta_0(J \setminus U) + \theta_0(J \setminus A) + \varepsilon$$

and it follows that

$$\theta_0(A) - \varepsilon < \theta_0(J \setminus U).$$

Since $J \setminus U$ is a closed bounded set contained in $A$ we conclude that

$$\theta_0(A) \leq \sup_{\substack{K \subseteq A \\ K \text{ closed bounded}}} \theta_0(K).$$

If $A \in \mathcal{M}(\theta_0)$ let $A_n = A \cap [-n, n]$, $n \in \mathbf{N}_+$. Then given $\varepsilon > 0$ and $n \in \mathbf{N}_+$, let $K_n$ be a closed bounded subset of $A_n$ such that $\theta_0(K_n) > \theta_0(A_n) - \varepsilon$. Clearly, there is no loss of generality to assume that $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ and by letting $n$ tend to plus infinity we get

$$\lim_{n \to \infty} \theta_0(K_n) \geq \theta_0(A) - \varepsilon.$$

Hence

$$\theta_0(A) = \sup_{\substack{K \subseteq A \\ K \text{ compact}}} \theta_0(K).$$

and Theorem 1.5.4 is completely proved.

**Theorem 1.5.5.** *Lebesgue measure $m_1$ equals the restriction of $\theta_0$ to $\mathcal{M}(\theta_0)$.*

PROOF. Recall that linear measure $v_1$ equals the restriction of $\theta_0$ to $\mathcal{R}$ and $m_1 = \bar{v}_1$. First suppose $E \in \mathcal{R}^-$ and choose $A, B \in \mathcal{R}$ such that $A \subseteq E \subseteq B$ and $B \setminus A \in \mathcal{Z}_{v_1}$. But then $\theta_0(E \setminus A) = 0$ and $E = A \cup (E \setminus A) \in \mathcal{M}(\theta_0)$ since the Carathéodory Theorem gives us a complete measure. Hence $m_1(E) = v_1(A) = \theta_0(E)$.

Conversely suppose $E \in \mathcal{M}(\theta_0)$. We will prove that $E \in \mathcal{R}^-$ and $m_1(E) = \theta_0(E)$. First assume that $E$ is bounded. Then for each positive integer $n$ there exist open $U_n \supseteq E$ and closed bounded $K_n \subseteq E$ such that

$$\theta_0(U_n) < \theta_0(E) + 2^{-n}$$

and

$$\theta_0(K_n) > \theta_0(E) - 2^{-n}.$$

The definitions yield $A = \cup_1^\infty K_n$, $B = \cap_1^\infty U_n \in \mathcal{R}$ and

$$\theta_0(E) = \theta_0(A) = \theta_0(B) = v_1(A) = v_1(B) = m_1(E).$$

It follows that $E \in \mathcal{R}^-$ and $\theta_0(E) = m_1(E)$.

In the general case set $E_n = E \cap [-n, n]$, $n \in \mathbf{N}_+$. Then from the above $E_n \in \mathcal{R}^-$ and $\theta_0(E_n) = m_1(E_n)$ for each $n$ and Theorem 1.5.5 follows by letting $n$ go to infinity.

The Carathéodory Theorem can be used to show the existence of volume measure on $\mathbf{R}^n$ but we do not go into this here since its existence follows by several other means below. By passing, let us note that the Carathéodory Theorem is very efficient to prove the existence of so called Haussdorff measures (see e.g. $[F]$), which are of great interest in Geometric Measure Theory.

**Exercises**

1. Prove that a subset $K$ of $\mathbf{R}$ is compact if and only if $K$ is closed and bounded.

2. Suppose $A \in \mathcal{R}^-$ and $m(A) < \infty$. Set $f(x) = m(A \cap ]-\infty, x])$, $x \in \mathbf{R}$. Prove that $f$ is continuous.

3. Suppose $A \in \mathcal{Z}_m$ and $B = \{x^3; x \in A\}$. Prove that $B \in \mathcal{Z}_m$.

4. Let $A$ be the set of all real numbers $x$ such that

$$| x - \frac{p}{q} | \leq \frac{1}{q^3}$$

for infinitely many pairs of positive integers $p$ and $q$. Prove that $A \in \mathcal{Z}_m$.

5. Let $I_1, ..., I_n$ be open subintervals of $\mathbf{R}$ such that

$$\mathbf{Q} \cap [0, 1] \subseteq \cup_{k=1}^{n} I_k.$$

Prove that $\Sigma_{k=1}^{n} m(I_k) \geq 1$.

6. If $E \in \mathcal{R}^-$ and $m(E) > 0$, for every $\alpha \in ]0, 1[$ there is an interval $I$ such that $m(E \cap I) > \alpha m(I)$. (Hint: $m(E) = \inf \Sigma_{k=1}^{\infty} m(I_k)$, where the infimum is taken over all intervals such that $\cup_{k=1}^{\infty} I_k \supseteq E$.)

7. If $E \in \mathcal{R}^-$ and $m(E) > 0$, then the set $E - E = \{x - y; x, y \in E\}$ contains an open non-empty interval centred at 0.(Hint: Take an interval $I$ with $m(E \cap I) \geq \frac{3}{4} m(I)$. Set $\varepsilon = \frac{1}{2} m(I)$. If $| x | \leq \varepsilon$, then $(E \cap I) \cap (x + (E \cap I)) \neq \phi$.)

8. Let $\mu$ be the restriction of the positive measure $\Sigma_{k=1}^{\infty} \delta_{\mathbf{R}, \frac{1}{k}}$ to $\mathcal{R}$. Prove that

$$\inf_{\substack{U \supseteq A \\ U \text{ open}}} \mu(U) > \mu(A)$$

if $A = \{0\}$.

## 1.6. Positive Measures Induced by Increasing Right Continuous Functions

Suppose $F : \mathbf{R} \to [0, \infty[$ is a right continuous increasing function such that

$$\lim_{x \to -\infty} F(x) = 0.$$

Set

$$L = \lim_{x \to \infty} F(x).$$

We will prove that there exists a unique positive measure $\mu : \mathcal{R} \to [0, L]$ such that

$$\mu(]-\infty, x]) = F(x), \ \ x \in \mathbf{R}.$$

The special case $L = 0$ is trivial so let us assume $L > 0$ and introduce

$$H(y) = \inf \{x \in \mathbf{R}; F(x) \geq y\}, \ \ 0 < y < L.$$

The definition implies that the function $H$ increases.

Suppose $a$ is a fixed real number. We claim that

$$\{y \in ]0, L[\,; \ \ H(y) \leq a\} = ]0, F(a)] \cap ]0, L[\,.$$

To prove this first suppose that $y \in ]0, L[$ and $H(y) \leq a$. Then to each positive integer $n$, there is an $x_n \in [H(y), H(y) + 2^{-n}[$ such that $F(x_n) \geq y$. Then $x_n \to H(y)$ as $n \to \infty$ and we obtain that $F(H(y)) \geq y$ since $F$ is right continuous. Thus, remembering that $F$ increases, $F(a) \geq y$. On the other hand, if $0 < y < L$ and $0 < y \leq F(a)$, then, by the very definition of $H(y)$, $H(y) \leq a$.

We now define

$$\mu = H(v_{1|]0,L[})$$

and get

$$\mu(]-\infty, x]) = F(x), \ \ x \in \mathbf{R}.$$

The uniqueness follows at once from Theorem 1.2.3. Note that the measure $\mu$ is a probability measure if $L = 1$.

**Exercises**

1. Suppose $F : \mathbf{R} \to \mathbf{R}$ is a right continuous increasing function. Prove that there is a unique positive measure $\mu$ on $\mathcal{R}$ such that

$$\mu(]a, x]) = F(x) - F(a), \text{ if } a, x \in \mathbf{R} \text{ and } a < x.$$

2. Suppose $F : \mathbf{R} \to \mathbf{R}$ is an increasing function. Prove that the set of all discontinuity points of $F$ is at most denumerable. (Hint: Assume first that $F$ is bounded and prove that the set of all points $x \in \mathbf{R}$ such that $F(x+) - F(x-) > \varepsilon$ is finite for every $\varepsilon > 0$.)

3. Suppose $\mu$ is a $\sigma$-finite positive measure on $\mathcal{R}$. Prove that the set of all $x \in \mathbf{R}$ such that $\mu(\{x\}) > 0$ is at most denumerable.

4. Suppose $\mu$ is a $\sigma$-finite positive measure on $\mathcal{R}_n$. Prove that there is an at most denumerable set of hyperplanes of the type

$$x_k = c \qquad (k = 1, ..., n,\ c \in \mathbf{R})$$

with positive $\mu$-measure.

5. Construct an increasing function $f : \mathbf{R} \to \mathbf{R}$ such that the set of discontinuity points of $f$ equals $\mathbf{Q}$.