CHAPTER 2

INTEGRATION

Introduction

In this chapter Lebesgue integration in abstract positive measure spaces is introduced. A series of famous theorems and lemmas will be proved.

2.1. Integration of Functions with Values in $[0, \infty]$

Recall that $[0, \infty] = [0, \infty[\cup \{\infty\}$. A subinterval of $[0, \infty]$ is defined in the natural way. We denote by $\mathcal{R}_{0,\infty}$ the σ-algebra generated by all subintervals of $[0, \infty]$. The class of all intervals of the type $]\alpha, \infty]$, $0 \leq \alpha < \infty$, (or of the type $[\alpha, \infty]$, $0 \leq \alpha < \infty$) generates the σ-algebra $\mathcal{R}_{0,\infty}$ and we get the following

Theorem 2.1.1. *Let* $(X, \mathcal{M})$ *be a measurable space and suppose* $f : X \to [0, \infty]$.
(a) *The function* f *is* $(\mathcal{M}, \mathcal{R}_{0,\infty})$*-measurable if* $f^{-1}(]\alpha, \infty]) \in \mathcal{M}$ *for every* $0 \leq \alpha < \infty$.
(b) *The function* f *is* $(\mathcal{M}, \mathcal{R}_{0,\infty})$*-measurable if* $f^{-1}([\alpha, \infty]) \in \mathcal{M}$ *for every* $0 \leq \alpha < \infty$.

Note that the set $\{f > \alpha\} \in \mathcal{M}$ for all real α if f is $(\mathcal{M}, \mathcal{R}_{0,\infty})$-measurable.
If $f, g : X \to [0, \infty]$ are $(\mathcal{M}, \mathcal{R}_{0,\infty})$-measurable, then $\min(f, g)$, $\max(f, g)$, and $f + g$ are $(\mathcal{M}, \mathcal{R}_{0,\infty})$-measurable, since, for each $\alpha \in [0, \infty[$,

$$\min(f, g) \geq \alpha \Leftrightarrow (f \geq \alpha \text{ and } g \geq \alpha)$$

$$\max(f, g) \geq \alpha \Leftrightarrow (f \geq \alpha \text{ or } g \geq \alpha)$$

and

$$\{f+g>\alpha\} = \bigcup_{q\in\mathbf{Q}} (\{f>\alpha-q\}\cap\{g>q\}).$$

Given functions $f_n : X \to [0,\infty]$, $n = 1,2,...$, $f = \sup_{n\geq 1} f_n$ is defined by the equation

$$f(x) = \sup\{f_n(x);\ n = 1,2,...\}.$$

Note that

$$f^{-1}(]\alpha,\infty]) = \cup_{n=1}^{\infty} f_n^{-1}(]\alpha,\infty])$$

for every real $\alpha \geq 0$ and, accordingly from this, the function $\sup_{n\geq 1} f_n$ is $(\mathcal{M},\mathcal{R}_{0,\infty})$-measurable if each f_n is $(\mathcal{M},\mathcal{R}_{0,\infty})$-measurable. Moreover, $f = \inf_{n\geq 1} f_n$ is given by

$$f(x) = \inf\{f_n(x);\ n = 1,2,...\}.$$

Since

$$f^{-1}([0,\alpha[) = \cup_{n=1}^{\infty} f_n^{-1}([0,\alpha[)$$

for every real $\alpha \geq 0$ we conclude that the function $f = \inf_{n\geq 1} f_n$ is $(\mathcal{M},\mathcal{R}_{0,\infty})$-measurable if each f_n is $(\mathcal{M},\mathcal{R}_{0,\infty})$-measurable.

Below we write

$$f_n \uparrow f$$

if f_n, $n = 1,2,...$, and f are functions from X into $[0,\infty]$ such that $f_n \leq f_{n+1}$ for each n and $f_n(x) \to f(x)$ for each $x \in X$ as $n \to \infty$.

An $(\mathcal{M},\mathcal{R}_{0,\infty})$-measurable function $\varphi : X \to [0,\infty]$ is called a simple measurable function if $\varphi(X)$ is a finite subset of $[0,\infty[$. If it is neccessary to be more precise, we say that φ is a simple $\mathcal{M}$-measurable function.

Theorem 2.1.2. *Let $f : X \to [0,\infty]$ be $(\mathcal{M},\mathcal{R}_{0,\infty})$-measurable. There exist simple measurable functions φ_n, $n \in \mathbf{N}_+$, on X such that $\varphi_n \uparrow f$.*

PROOF. Given $n \in \mathbf{N}_+$, set

$$E_{in} = f^{-1}(\left[\frac{i-1}{2^n},\frac{i}{2^n}\right[),\ i \in \mathbf{N}_+$$

and

$$\rho_n = \sum_{i=1}^{\infty} \frac{i-1}{2^n} \chi_{E_{in}} + \infty \chi_{f^{-1}(\{\infty\})}.$$

It is obvious that $\rho_n \leq f$ and that $\rho_n \leq \rho_{n+1}$. Now set $\varphi_n = \min(n, \rho_n)$ and we are done.

Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $\varphi : X \to [0, \infty[$ a simple measurable function. If $\alpha_1, ..., \alpha_n$ are the distinct values of the simple function φ, and if $E_i = \varphi^{-1}(\{\alpha_i\})$, $i = 1, ..., n$, then

$$\varphi = \Sigma_{i=1}^n \alpha_i \chi_{E_i}.$$

Furthermore, if $A \in \mathcal{M}$ we define

$$\nu(A) = \int_A \varphi d\mu = \Sigma_{k=1}^n \alpha_i \mu(E_i \cap A) = \Sigma_{k=1}^n \alpha_i \mu^{E_i}(A).$$

Clearly, ν is a positive measure since each term in the right side is a positive measure as a function of A. Note that

$$\int_A \alpha \varphi d\mu = \alpha \int_A \varphi d\mu \text{ if } 0 \leq \alpha < \infty$$

and

$$\int_A \varrho d\mu = a\mu(A)$$

if $a \in [0, \infty[$ and ϱ is a simple measurable function such that $\varrho = a$ on A.

If ψ is another simple measurable function and $\varphi \leq \psi$,

$$\int_A \varphi d\mu \leq \int_A \psi d\mu.$$

To see this, let $\beta_1, ..., \beta_p$ be the distinct values of ψ and $F_j = \psi^{-1}(\{\beta_j\})$, $j = 1, ..., p$. Now, putting $B_{ij} = E_i \cap F_j$,

$$\int_A \varphi d\mu = \nu(\cup_{ij}(A \cap B_{ij}))$$

$$= \Sigma_{ij} \nu(A \cap B_{ij}) = \Sigma_{ij} \int_{A \cap B_{ij}} \varphi d\mu = \Sigma_{ij} \int_{A \cap B_{ij}} \alpha_i d\mu$$

$$\leq \Sigma_{ij} \int_{A \cap B_{ij}} \beta_j d\mu = \int_A \psi d\mu.$$

In a similar way one proves that

$$\int_A (\varphi + \psi) d\mu = \int_A \varphi d\mu + \int_A \psi d\mu.$$

From the above it follows that

$$\int_A \varphi \chi_A d\mu = \int_A \Sigma_{i=1}^n \alpha_i \chi_{E_i \cap A} d\mu$$

$$= \Sigma_{i=1}^n \alpha_i \int_A \chi_{E_i \cap A} d\mu = \Sigma_{i=1}^n \alpha_i \mu(E_i \cap A)$$

and

$$\int_A \varphi \chi_A d\mu = \int_A \varphi d\mu.$$

If $f : X \to [0, \infty]$ is an $(\mathcal{M}, \mathcal{R}_{0,\infty})$-measurable function and $A \in \mathcal{M}$, we define

$$\int_A f d\mu = \sup \left\{ \int_A \varphi d\mu;\ 0 \leq \varphi \leq f,\ \varphi \text{ simple measurable} \right\}$$

$$= \sup \left\{ \int_A \varphi d\mu;\ 0 \leq \varphi \leq f,\ \varphi \text{ simple measurable and } \varphi = 0 \text{ on } A^c \right\}.$$

The left member in this equation is called the Lebesgue integral of f over A with respect to the measure μ. Sometimes we also speek of the μ-integral of f over A. The two definitions of the μ-integral of a simple measurable function $\varphi : X \to [0, \infty[$ over A agree.

From now on in this section, an $(\mathcal{M}, \mathcal{R}_{0,\infty})$-measurable function $f : X \to [0, \infty]$ is simply called measurable.

The following properties are immediate consequences of the definitions. The functions and sets occurring in the equations are assumed to be measurable.

(a) If f, $g \geq 0$ and $f \leq g$ on A, then $\int_A f d\mu \leq \int_A g d\mu$.

(b) $\int_A f d\mu = \int_X \chi_A f d\mu$.

(c) If $f \geq 0$ and $\alpha \in [0, \infty[$, then $\int_A \alpha f d\mu = \alpha \int_A f d\mu$.

(d) $\int_A f d\mu = 0$ if $f = 0$ and $\mu(A) = \infty$.

(e) $\int_A f d\mu = 0$ if $f = \infty$ and $\mu(A) = 0$.

If $f : X \to [0, \infty]$ is measurable and $0 < \alpha < \infty$, then $f \geq \alpha\chi_{f^{-1}([\alpha,\infty])} = \alpha\chi_{\{f\geq\alpha\}}$and

$$\int_X f d\mu \geq \int_X \alpha\chi_{\{f\geq\alpha\}} d\mu = \alpha \int_X \chi_{\{f\geq\alpha\}} d\mu.$$

This proves the so called Markov Inequality

$$\mu(f \geq \alpha) \leq \frac{1}{\alpha} \int_X f d\mu$$

where we write $\mu(f \geq \alpha)$ instead of the more precise expression $\mu(\{f \geq \alpha\})$.

Example 2.1.1. Suppose $f : X \to [0, \infty]$ is measurable and

$$\int_X f d\mu < \infty.$$

We claim that

$$\{f = \infty\} = f^{-1}(\{\infty\}) \in \mathcal{Z}_\mu.$$

To prove this we use the Markov Inequality and have

$$\mu(f = \infty) \leq \mu(f \geq \alpha) \leq \frac{1}{\alpha} \int_X f d\mu$$

for each $\alpha \in]0, \infty[$. Thus $\mu(f = \infty) = 0$.

Example 2.1.2. Suppose $f : X \to [0, \infty]$ is measurable and

$$\int_X f d\mu = 0.$$

We claim that

$$\{f > 0\} = f^{-1}(]0, \infty]) \in \mathcal{Z}_\mu.$$

To see this, note that

$$f^{-1}(]0, \infty]) = \cup_{n=1}^{\infty} f^{-1}(\left]\frac{1}{n}, \infty\right])$$

Furthermore, for every fixed $n \in \mathbf{N}_+$, the Markov Inequality yields

$$\mu(f > \frac{1}{n}) \leq n \int_X f d\mu = 0$$

and we get $\{f > 0\} \in \mathcal{Z}_\mu$ since a countable union of null sets is a null set.

We now come to one of the most important results in the theory.

Theorem 2.1.3. (Lebesgue's Monotone Convergence Theorem) *Let* $f_n : X \to [0, \infty]$, $n = 1, 2, 3, ...,$ *be a sequence of measurable functions and suppose that* $f_n \uparrow f$, that is $0 \leq f_1 \leq f_2 \leq ...$ *and*

$$f_n(x) \to f(x) \text{ as } n \to \infty, \text{ for every } x \in X.$$

Then f *is measurable and*

$$\int_X f_n d\mu \to \int_X f d\mu \text{ as } n \to \infty.$$

PROOF. The function f is measurable since $f = \sup_{n \geq 1} f_n$.

The inequalities $f_n \leq f_{n+1} \leq f$ yield $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu \leq \int_X f d\mu$ and we conclude that there exists an $\alpha \in [0, \infty]$ such that

$$\int_X f_n d\mu \to \alpha \text{ as } n \to \infty$$

and

$$\alpha \leq \int_X f d\mu.$$

To prove the reverse inequality, let φ be any simple measurable function such that $0 \leq \varphi \leq f$, let $0 < \theta < 1$ be a constant, and define, for fixed $n \in \mathbf{N}_+$,

$$A_n = \{x \in X;\; f_n(x) \geq \theta\varphi(x)\}.$$

If $\alpha_1, ..., \alpha_p$ are the distinct values of φ,

$$A_n = \cup_{k=1}^{p} (\{x \in X;\; f_n(x) \geq \theta\alpha_k\} \cap \{\varphi = \alpha_k\})$$

and it follows that A_n is measurable. Clearly, $A_1 \subseteq A_2 \subseteq ...$. Moreover, if $f(x) = 0$, then $x \in A_1$ and if $f(x) > 0$, then $\theta\varphi(x) < f(x)$ and $x \in A_n$ for all sufficiently large n. Thus $\cup_{n=1}^{\infty} A_n = X$. Now

$$\alpha \geq \int_{A_n} f_n d\mu \geq \theta \int_{A_n} \varphi d\mu$$

and we get

$$\alpha \geq \theta \int_X \varphi d\mu$$

since the map $A \to \int_A \varphi d\mu$ is a positive measure on $\mathcal{M}$. By letting $\theta \uparrow 1$,

$$\alpha \geq \int_X \varphi d\mu$$

and, hence

$$\alpha \geq \int_X f d\mu.$$

The theorem follows.

Theorem 2.1.4. (a) *Let* $f, g : X \to [0, \infty]$ *be measurable functions. Then*

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

(b) **(Beppo Levi's Theorem)** *If* $f_k : X \to [0, \infty]$, $k = 1, 2, ...$ *are measurable,*

$$\int_X \Sigma_{k=1}^{\infty} f_k d\mu = \Sigma_{k=1}^{\infty} \int_X f_k d\mu$$

PROOF. (a) Let $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ be sequences of simple and measurable functions such that $0 \leq \varphi_n \uparrow f$ and $0 \leq \psi_n \uparrow g$. We proved above that

$$\int_X (\varphi_n + \psi_n) d\mu = \int_X \varphi_n d\mu + \int_X \psi_n d\mu$$

and, by letting $n \to \infty$, Part (a) follows from Lebesgue's Monotone Convergence Theorem.

(b) Part (a) and induction imply that

$$\int_X \Sigma_{k=1}^{n} f_k d\mu = \Sigma_{k=1}^{n} \int_X f_k d\mu$$

and the result follows from monotone convergence.

Theorem 2.1.5. *Suppose* $w : X \to [0, \infty]$ *is a measurable function and define*

$$\nu(A) = \int_A w d\mu, \ A \in \mathcal{M}.$$

Then ν *is a positive measure and*

$$\int_A f d\nu = \int_A f w d\mu, \ A \in \mathcal{M}$$

for every measurable function $f : X \to [0, \infty]$.

PROOF. Clearly, $\nu(\phi) = 0$. Suppose $(E_k)_{k=1}^{\infty}$ is a disjoint denumerable collection of members of $\mathcal{M}$ and set $E = \cup_{k=1}^{\infty} E_k$. Then

$$\nu(\cup_{k=1}^{\infty} E_k) = \int_E w d\mu = \int_X \chi_E w d\mu = \int_X \Sigma_{k=1}^{\infty} \chi_{E_k} w d\mu$$

where, by the Beppo Levi Theorem, the right member equals

$$\Sigma_{k=1}^{\infty} \int_X \chi_{E_k} w d\mu = \Sigma_{k=1}^{\infty} \int_{E_k} w d\mu = \Sigma_{k=1}^{\infty} \nu(E_k).$$

This proves that ν is a positive measure.

Let $A \in \mathcal{M}$. To prove the last part in Theorem 2.1.5 we introduce the class $\mathcal{C}$ of all measurable functions $f : X \to [0, \infty]$ such that

$$\int_A f d\nu = \int_A f w d\mu.$$

The indicator function of a measurable set belongs to $\mathcal{C}$ and from this we conclude that every simple measurable function belongs to $\mathcal{C}$. Furthermore, if $f_n \in \mathcal{C}$, $n \in \mathbf{N}$, and $f_n \uparrow f$, the Lebesgue Monotone Convergence Theorem proves that $f \in \mathcal{C}$. Thus in view of Theorem 2.1.2 the class $\mathcal{C}$ contains every measurable function $f : X \to [0, \infty]$. This completes the proof of Theorem 2.1.5.

The measure ν in Theorem 2.1.5 is written

$$\nu = w\mu$$

or

$$d\nu = w d\mu.$$

Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence in $[-\infty, \infty]$. First put $\beta_k = \inf \{\alpha_k, \alpha_{k+1}, \alpha_{k+2}, ...\}$ and $\gamma = \sup \{\beta_1, \beta_2, \beta_3, ..\} = \lim_{n\to\infty} \beta_n$. We call γ the lower limit of $(\alpha_n)_{n=1}^{\infty}$ and write

$$\gamma = \liminf_{n\to\infty} \alpha_n.$$

Note that

$$\gamma = \lim_{n\to\infty} \alpha_n$$

if the limit exists. Now put $\beta_k = \sup \{\alpha_k, \alpha_{k+1}, \alpha_{k+2}, ...\}$ and $\gamma = \inf \{\beta_1, \beta_2, \beta_3, ..\} = \lim_{n\to\infty} \beta_n$. We call γ the upper limit of $(\alpha_n)_{n=1}^{\infty}$ and write

$$\gamma = \limsup_{n\to\infty} \alpha_n.$$

Note that

$$\gamma = \lim_{n\to\infty} \alpha_n$$

if the limit exists.

Given measurable functions $f_n : X \to [0, \infty]$, $n = 1, 2, ...$, the function $\liminf_{n\to\infty} f_n$ is measurable. In particular, if

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$, then f is measurable.

Theorem 2.1.6. (Fatou's Lemma) *If* $f_n : X \to [0, \infty]$, $n = 1, 2, ...$, *are measurable*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

PROOF. Introduce

$$g_k = \inf_{n\geq k} f_n.$$

The definition gives that $g_k \uparrow \liminf_{n\to\infty} f_n$ and, moreover,

$$\int_X g_k d\mu \leq \int_X f_n d\mu, \ n \geq k$$

and

$$\int_X g_k d\mu \leq \inf_{n\geq k} \int_X f_n d\mu.$$

The Fatou Lemma now follows by monotone convergence.

Below we often write

$$\int_E f(x) d\mu(x)$$

instead of

$$\int_E f d\mu.$$

Example 2.1.3. Suppose $a \in \mathbf{R}$ and $f : (\mathbf{R}, \mathcal{R}^-) \to ([0, \infty], \mathcal{R}_{0,\infty})$ is measurable. We claim that

$$\int_{\mathbf{R}} f(x + a) dm(x) = \int_{\mathbf{R}} f(x) dm(x).$$

First if $f = \chi_A$, where $A \in \mathcal{R}^-$,

$$\int_{\mathbf{R}} f(x+a)dm(x) = \int_{\mathbf{R}} \chi_{A-a}(x)dm(x) = m(A-a) =$$

$$m(A) = \int_{\mathbf{R}} f(x)dm(x).$$

Next it is clear that the relation we want to prove is true for simple measurable functions and finally, we use the Lebesgue Dominated Convergence Theorem to deduce the general case.

Exercises

1. Suppose $f_n : X \to [0,\infty]\,,\ n = 1, 2, ...,$ are measurable and

$$\Sigma_{k=1}^{\infty}\mu(f_n > 1) < \infty.$$

Prove that

$$\left\{\limsup_{n\to\infty} f_n > 1\right\} \in \mathcal{Z}_\mu\ .$$

2. Set $\ f_n = n^2\chi_{\left[0,\frac{1}{n}\right]}$, $n \in \mathbf{N}_+$. Prove that

$$\int_{\mathbf{R}} \liminf_{n\to\infty} f_n dm = 0 < \infty = \liminf_{n\to\infty} \int_{\mathbf{R}} f_n dm$$

(the inequality in the Fatou Lemma may be strict).

3. Suppose $f : (\mathbf{R},\mathcal{R}^-) \to ([0,\infty]\,,\mathcal{R}_{0,\infty})$ is measurable and set

$$g(x) = \Sigma_{k=1}^{\infty} f(x+k),\ \ x \in \mathbf{R}.$$

Show that

$$\int_{\mathbf{R}} gdm < \infty \text{ if and only if } \{f > 0\} \in \mathcal{Z}_m.$$

4. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $f : X \to [0, \infty]$ an $(\mathcal{M}, \mathcal{R}_{0,\infty})$-measurable function such that

$$f(X) \subseteq \mathbf{N}$$

and

$$\int_X f d\mu < \infty.$$

For every $t \geq 0$, set

$$F(t) = \mu(f > t) \text{ and } G(t) = \mu(f \geq t).$$

Prove that

$$\int_X f d\mu = \Sigma_{n=0}^{\infty} F(n) = \Sigma_{n=1}^{\infty} G(n).$$

2.2. Integration of Functions with Arbitrary Sign

As usual suppose $(X, \mathcal{M}, \mu)$ is a positive measure space. In this section when we speak of a measurable function $f : X \to \mathbf{R}$ it is understood that f is an $(\mathcal{M}, \mathcal{R})$-measurable function, if not otherwise stated. If $f, g : X \to \mathbf{R}$ are measurable, the sum $f + g$ is measurable since

$$\{f + g > \alpha\} = \bigcup_{q \in \mathbf{Q}} (\{f > \alpha - q\} \cap \{g > q\})$$

for each real α. Besides the function $-f$ and the difference $f - g$ are measurable. It follows that a function $f : X \to \mathbf{R}$ is measurable if and only if the functions $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$ are measurable since $f = f^+ - f^-$.

We write $f \in \mathcal{L}^1(\mu)$ if $f : X \to \mathbf{R}$ is measurable and

$$\int_X \mid f \mid d\mu < \infty$$

and in this case we define

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

Note that

$$| \int_X f d\mu | \leq \int_X | f | d\mu$$

since $| f |= f^+ + f^-$. Moreover defining

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu, \text{ all } E \in \mathcal{M}$$

it follows that

$$\int_E f d\mu = \int_X \chi_E f d\mu.$$

Note that

$$\int_E f d\mu = 0 \text{ if } \mu(E) = 0.$$

Sometimes we write

$$\int_E f(x) d\mu(x)$$

instead of

$$\int_E f d\mu.$$

If $f, g \in \mathcal{L}^1(\mu)$, setting $h = f + g$,

$$\int_X | h | d\mu \leq \int_X | f | d\mu + \int_X | g | d\mu < \infty$$

and it follows that $h + g \in \mathcal{L}^1(\mu)$. Moreover,

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$

and the equation

$$h^+ + f^- + g^- = f^+ + g^+ + h^-$$

gives

$$\int_X h^+ d\mu + \int_X f^- d\mu + \int_X g^- d\mu = \int_X f^+ d\mu + \int_X g^+ d\mu + \int_X h^- d\mu.$$

Thus

$$\int_X h d\mu = \int_X f d\mu + \int_X g d\mu.$$

Moreover,

$$\int_X \alpha f d\mu = \alpha \int_X f d\mu$$

for each real α. The case $\alpha \geq 0$ follows from (c) in Section 2.1. The case $\alpha = -1$ is also simple since $(-f)^+ = f^-$ and $(-f)^- = f^+$.

Theorem 2.2.1. (Lebesgue's Dominated Convergence Theorem) S*uppose* $f_n : X \to \mathbf{R}$, $n = 1, 2, ...,$ *are measurable and*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. *Moreover, suppose there exists a function* $g \in \mathcal{L}^1(\mu)$ *such that*

$$\mid f_n(x) \mid \leq g(x), \text{ all } x \in X \text{ and } n \in \mathbf{N}_+.$$

Then $f \in \mathcal{L}^1(\mu)$,

$$\lim_{n\to\infty} \int_X \mid f_n - f \mid d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. Since $\mid f \mid \leq g$, the function f is real-valued and measurable since f^+ and f^- are measurable. Note here that

$$f^{\pm}(x) = \lim_{n\to\infty} f_n^{\pm}(x), \text{all } x \in X.$$

We now apply the Fatous Lemma to the functions $2g- \mid f_n - f \mid, n = 1, 2, ...,$ and have

$$\int_X 2g d\mu \leq \liminf_{n\to\infty} \int_X (2g- \mid f_n - f \mid) d\mu$$

$$= \int_X 2g d\mu - \limsup_{n\to\infty} \int_X \mid f_n - f \mid d\mu.$$

But $\int_X 2g d\mu$ is finite and we get

$$\lim_{n\to\infty} \int_X \mid f_n - f \mid d\mu = 0.$$

Since

$$| \int_X f_n d\mu - \int_X f d\mu |=| \int_X (f - f_n) d\mu |\leq \int_X | f - f_n | d\mu$$

the last part in Theorem 2.2.1 follows from the first part. The theorem is proved.

Example 2.2.1. Suppose $f :]a, b[\times X \to \mathbf{R}$ is a function such that $f(t, \cdot) \in \mathcal{L}^1(\mu)$ for each $t \in]a, b[$ and, moreover, assume $\frac{\partial f}{\partial t}$ exists and

$$| \frac{\partial f}{\partial t}(t, x) |\leq g(x) \text{ for all } (t, x) \in]a, b[\times X$$

where $g \in \mathcal{L}^1(\mu)$. Set

$$F(t) = \int_X f(t, x) d\mu(x) \text{ if } t \in]a, b[\,.$$

We claim that F is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(t, x) d\mu(x).$$

To see this let $t_* \in]a, b[$ be fixed and choose a sequence $(t_n)_{n=1}^{\infty}$ in $]a, b[\setminus \{t_*\}$ which converges to t_*. Define

$$h_n(x) = \frac{f(t_n, x) - f(t_*, x)}{t_n - t_*} \text{ if } x \in X.$$

Here each h_n is measurable and

$$\lim_{n\to\infty} h_n(x) = \frac{\partial f}{\partial t}(t_*, x) \text{ for all } x \in X.$$

Furthermore, for each fixed n and x there is a $\tau_{n,x} \in]t_n, t_*[$ such that $h_n(x) = \frac{\partial f}{\partial t}(\tau_{n,x}, x)$ and we conclude that $| h_n(x) |\leq g(x)$ for every $x \in X$. Since

$$\frac{F(t_n) - F(t_*)}{t_n - t_*} = \int_X h_n(x) d\mu(x)$$

the claim above now follows from the Lebesgue Dominated Convergence Theorem.

Suppose $S(x)$ is a statement, which depends on $x \in X$. We will say that $S(x)$ holds almost (or μ-almost) everywhere if there exists an $N \in \mathcal{Z}_\mu$ such that $S(x)$ holds at every point of $X \setminus N$. In this case we write "S holds a.e. " or "S holds a.e. $[\mu]$". Sometimes we prefer to write "$S(x)$ holds a.e." or "$S(x)$ holds a.e. $[\mu]$". If the underlying measure space is a probability space, we often say "almost surely" instead of almost everywhere. The term "almost surely" is abbreviated a.s.

Suppose $f : X \to \mathbf{R}$, is an $(\mathcal{M}, \mathcal{R})$-measurable functions and $g : X \to \mathbf{R}$. If $f = g$ a.e. $[\mu]$ there exists an $N \in \mathcal{Z}_\mu$ such that $f(x) = g(x)$ for every $x \in X \setminus N$. We claim that g is $(\mathcal{M}^-, \mathcal{R})$-measurable. To see this let $\alpha \in \mathbf{R}$ and use that

$$\{g > \alpha\} = [\{f > \alpha\} \cap (X \setminus N)] \cup [\{g > \alpha\} \cap N] .$$

Now if we define

$$A = \{f > \alpha\} \cap (X \setminus N)$$

the set $A \in \mathcal{M}$ and

$$A \subseteq \{g > \alpha\} \subseteq A \cup N.$$

Accordingly from this $\{g > \alpha\} \in \mathcal{M}^-$ and g is $(\mathcal{M}^-, \mathcal{R})$-measurable since α is an arbitrary real number.

Next suppose $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}_+$, is a sequence of $(\mathcal{M}, \mathcal{R})$-measurable functions and $f : X \to \mathbf{R}$ a function. Recall if

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ all } x \in X$$

then f is $(\mathcal{M}, \mathcal{R})$-measurable since

$$\{f > \alpha\} = \cup_{k,l \in \mathbf{N}_+} \cap_{n \geq k} \left\{f_n > \alpha + l^{-1}\right\}, \text{ all } \alpha \in \mathbf{R}.$$

If we only assume that

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ a.e. } [\mu]$$

then f need not be $(\mathcal{M}, \mathcal{R})$-measurable but f is $(\mathcal{M}^-, \mathcal{R})$-measurable. To see this suppose $N \in \mathcal{Z}_\mu$ and

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ all } x \in X \setminus N.$$

Then

$$\lim_{n\to\infty} \chi_{X\setminus N}(x) f_n(x) = \chi_{X\setminus N}(x) f(x)$$

and it follows that the function $\chi_{X\setminus N} f$ is $(\mathcal{M}, \mathcal{R})$-measurable. Since $f = \chi_{X\setminus N} f$ a.e. $[\mu]$ it follows that f is $(\mathcal{M}^-, \mathcal{R})$-measurable. The next example shows that f need not be $(\mathcal{M}, \mathcal{R})$-measurable.

Example 2.2.2. Let $X = \{0, 1, 2\}$, $\mathcal{M} = \{\phi, \{0\}, \{1, 2\}, X\}$, and $\mu(A) = \chi_A(0)$, $A \in \mathcal{M}$. Set $f_n = \chi_{\{1,2\}}$, $n \in \mathbf{N}_+$, and $f(x) = x$, $x \in X$. Then each f_n is $(\mathcal{M}, \mathcal{R})$-measurable and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e. } [\mu]$$

since

$$\left\{ x \in X;\ \lim_{n\to\infty} f_n(x) = f(x) \right\} = \{0, 1\}$$

and $N = \{1, 2\}$ is a μ-null set. The function f is not $(\mathcal{M}, \mathcal{R})$-measurable.

Suppose $f, g \in \mathcal{L}^1(\mu)$. The functions f and g are equal almost everywhere with respect to μ if and only if $\{f \neq g\} \in \mathcal{Z}_\mu$. This is easily seen to be an equivalence relation and the set of all equivalence classes is denoted by $L^1(\mu)$. Moreover, if $f = g$ a.e. $[\mu]$, then

$$\int_X f d\mu = \int_X g d\mu$$

since

$$\int_X f d\mu = \int_{\{f=g\}} f d\mu + \int_{\{f\neq g\}} f d\mu = \int_{\{f=g\}} f d\mu = \int_{\{f=g\}} g d\mu$$

and, in a similar way,

$$\int_X g d\mu = \int_{\{f=g\}} g d\mu.$$

Below we consider the elements of $L^1(\mu)$ as members of $\mathcal{L}^1(\mu)$ and two members of $L^1(\mu)$ are identified if they are equal a.e. $[\mu]$. From this convention it is straight-forward to define $f+g$ and αf for all $f,g \in L^1(\mu)$ and $\alpha \in \mathbf{R}$. Moreover, we get

$$\int_X (f+g)d\mu = \int_X fd\mu + \int_X gd\mu \text{ if } f,g \in L^1(\mu)$$

and

$$\int_X \alpha f d\mu = \alpha \int_X fd\mu \text{ if } f \in L^1(\mu) \text{ and } \alpha \in \mathbf{R}.$$

Next we give two theorems where exceptional null sets enter. The first one is a mild variant of Theorem 2.2.1 and needs no proof.

Theorem 2.2.2. *Suppose $(X,\mathcal{M},\mu)$ is a positive complete measure space and let $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}_+$, be measurable functions such that*

$$\sup_{n\in\mathbf{N}_+} \mid f_n(x) \mid \leq g(x) \text{ a.e. } [\mu]$$

where $g \in L^1(\mu)$. Moreover, suppose $f : X \to \mathbf{R}$ is a function and

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ a.e. } [\mu].$$

T*hen,* $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X \mid f_n - f \mid d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X fd\mu.$$

Theorem 2.2.3. *Suppose $(X,\mathcal{M},\mu)$ is a positive measure space.*

(a) *If $f : (X,\mathcal{M}^-) \to ([0,\infty], \mathcal{R}_{0,\infty})$ is measurable there exists a measurable function $g : (X,\mathcal{M}) \to ([0,\infty], \mathcal{R}_{0,\infty})$ such that $f = g$* a.e. $[\mu]$.

(b) *If $f : (X,\mathcal{M}^-) \to (\mathbf{R},\mathcal{R})$ is measurable there exists a measurable function $g : (X,\mathcal{M}) \to (\mathbf{R},\mathcal{R})$ such that $f = g$* a.e. $[\mu]$.

PROOF. Since $f = f^+ - f^-$ it is enough to prove Part (a). There exist simple $\mathcal{M}^-$-measurable functions φ_n , $n \in \mathbf{N}_+$, such that $0 \leq \varphi_n \uparrow f$. For each fixed

n suppose $\alpha_{1n}, ..., \alpha_{k_n n}$ are the distinct values of φ_n and choose for each fixed $i = 1, ..., k_n$ a set $A_{in} \subseteq \varphi_n^{-1}(\{\alpha_{in}\})$ such that $A_{in} \in \mathcal{M}$ and $\varphi_n^{-1}(\alpha_{in}) \setminus A_{in} \in \mathcal{Z}_{\bar{\mu}}$. Set

$$\psi_n = \Sigma_{i=1}^{k_n} \alpha_{in} \chi_{A_{in}}.$$

Clearly $\psi_n(x) \uparrow f(x)$ if $x \in E =_{def} \cap_{n=1}^{\infty} (\cup_{i=1}^{k_n} A_{in})$ and $\mu(X \setminus E) = 0$. We now define $g(x) = f(x)$, if $x \in E$, and $g(x) = 0$ if $x \in X \setminus E$. The theorem is proved.

Exercises

1. Suppose f and g are real-valued measurable functions. Prove that f^2 and fg are measurable functions.

2. Suppose $f \in L^1(\mu)$. Prove that

$$\lim_{\alpha \to \infty} \int_{|f| \geq \alpha} | f | \, d\mu = 0.$$

(Here $\int_{|f| \geq \alpha}$ means $\int_{\{|f| \geq \alpha\}}$.)

3. Suppose $f \in L^1(\mu)$. Prove that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_E | f | \, d\mu < \varepsilon$$

whenever $\mu(E) < \delta$.

4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of $(\mathcal{M}, \mathcal{R})$-measurable functions. Prove that the set of all $x \in \mathbf{R}$ such that the sequence $(f_n(x))_{n=1}^{\infty}$ converges to a real limit belongs to $\mathcal{M}$.

5. Let $(X, \mathcal{M}, \mathcal{R})$ be a positive measure space such that $\mu(A) = 0$ or ∞ for every $A \in \mathcal{M}$. Show that $f \in L^1(\mu)$ if and only if $f(x) = 0$ a.e. $[\mu]$.

6. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and suppose f and g are non-negative measurable functions such that

$$\int_A f d\mu = \int_A g d\mu, \text{ all } A \in \mathcal{M}.$$

(a) Prove that $f = g$ a.e. $[\mu]$ if μ is σ-finite.
(b) Prove that the conclusion in Part (a) may fail if μ is not σ-finite.

7. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space and suppose the functions $f_n : X \to \mathbf{R}$, $n = 1, 2, ...$, are measurable. Show that there is a sequence $(\alpha_n)_{n=1}^{\infty}$ of positive real numbers such that

$$\lim_{n\to\infty} \alpha_n f_n = 0 \text{ a.e. } [\mu].$$

8. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and let $f_n : X \to \mathbf{R}$, $n = 1, 2, ...$, be a sequence in $L^1(\mu)$ which converges to f a.e. $[\mu]$ as $n \to \infty$. Suppose $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X \mid f_n \mid d\mu = \int_X \mid f \mid d\mu.$$

Show that

$$\lim_{n\to\infty} \int_X \mid f_n - f \mid d\mu = 0.$$

9. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space and suppose $f \in L^1(\mu)$ is a bounded function such that

$$\int_X f^2 d\mu = \int_X f^3 d\mu = \int_X f^4 d\mu.$$

Prove that $f = \chi_A$ for an appropriate $A \in \mathcal{M}$.

10. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space and $f : X \to \mathbf{R}$ a measurable function. Prove that $f \in L^1(\mu)$ if and only if

$$\Sigma_{k=1}^{\infty} \mu(\mid f \mid \geq k) < \infty.$$

11. Suppose $f \in L^1(m)$. Prove that the series $\Sigma_{k=-\infty}^{\infty} f(x+k)$ converges for m-almost all x.

12. a) Suppose $f : \mathbf{R} \rightarrow [0, \infty[$ is Lebesgue measurable and $\int_{\mathbf{R}} f dm < \infty$. Prove that

$$\lim_{\alpha \to \infty} \alpha m(f \geq \alpha) = 0.$$

b) Find a Lebesgue measurable function $f : \mathbf{R} \rightarrow [0, \infty[$ such that $f \notin L^1(m)$, $m(f > 0) < \infty$, and

$$\lim_{\alpha \to \infty} \alpha m(f \geq \alpha) = 0.$$

2.3 Comparison of Riemann and Lebesgue Integrals

In this section we will show that the Lebesgue integral is a natural generalization of the Riemann integral. For short, the discussion is restricted to a closed and bounded interval.

Let $[a, b]$ be a closed and bounded interval and suppose $f : [a, b] \rightarrow \mathbf{R}$ is a bounded function. For any partition

$$\Delta : a = x_0 < x_1 < ... < x_n = b$$

of $[a, b]$ define

$$S_\Delta f = \Sigma_{i=1}^{n} (\sup_{]x_{i-1}, x_i]} f)(x_i - x_{i-1})$$

and

$$s_\Delta f = \Sigma_{k=1}^{n} (\inf_{]x_{i-1}, x_i]} f)(x_i - x_{i-1}).$$

The function f is Riemann integrable if

$$\inf_{\Delta} S_\Delta f = \sup_{\Delta} s_\Delta f$$

and the Riemann integral $\int_a^b f(x) dx$ is, by definition, equal to this common value.

Below an $((\mathcal{R}^-)_{[a,b]}, \mathcal{R})$-measurable function is simply called Lebesgue measurable. Furthermore, we write m instead of $m_{|[a,b]}$.

Theorem 2.3.1. *A bounded function $f : [a,b] \to \mathbf{R}$ is Riemann integrable if and only if the set of discontinuity points of f is a Lebesgue null set. Moreover, if the set of discontinuity points of f is a Lebesgue null set, then* f is *Lebesgue measurable and*

$$\int_a^b f(x)dx = \int_{[a,b]} f dm.$$

PROOF. A partition $\Delta' : a = x'_0 < x'_1 < ... < x'_{n'} = b$ is a refinement of a partition $\Delta : a = x_0 < x_1 < ... < x_n = b$ if each x_k is equal to some x'_l and in this case we write $\Delta \prec \Delta'$. The definitions give $S_\Delta f \geq S_{\Delta'} f$ and $s_\Delta f \leq s_{\Delta'} f$ if $\Delta \prec \Delta'$. We define, $\text{mesh}(\Delta) = \max_{1 \leq i \leq n}(x_i - x_{i-1})$.

First suppose f is Riemann integrable. For each partition Δ let

$$G_\Delta = f(a)\chi_{\{a\}} + \Sigma_{i=1}^n (\sup_{]x_{i-1},x_i]} f)\chi_{]x_{i-1},x_i]}$$

and

$$g_\Delta = f(a)\chi_{\{a\}} + \Sigma_{i=1}^n (\inf_{]x_{i-1},x_i]} f)\chi_{]x_{i-1},x_i]}$$

and note that

$$\int_{[a,b]} G_\Delta dm = S_\Delta f$$

and

$$\int_{[a,b]} g_\Delta dm = s_\Delta f.$$

Suppose Δ_k, $k = 1, 2, ...$, is a sequence of partitions such that $\Delta_k \prec \Delta_{k+1}$,

$$S_{\Delta_k} f \downarrow \int_a^b f(x)dx$$

and

$$s_{\Delta_k} f \uparrow \int_a^b f(x)dx$$

as $k \to \infty$. Let $G = \lim_{k\to\infty} G_{\Delta_k}$ and $g = \lim_{k\to\infty} g_{\Delta_k}$. Then G and g are $(\mathcal{R}_{[a,b]}, \mathcal{R})$-measurable, $g \leq f \leq G$, and by dominated convergence

$$\int_{[a,b]} Gdm = \int_{[a,b]} gdm = \int_a^b f(x)dx.$$

But then

$$\int_{[a,b]} (G-g)dm = 0$$

so that $G = g$ a.e. $[m]$ and therefore $G = f$ a.e. $[m]$. In particular, f is Lebesgue measurable and

$$\int_a^b f(x)dx = \int_{[a,b]} fdm.$$

Set

$$N = \{x;\ g(x) < f(x) \text{ or } f(x) < G(x)\}.$$

We proved above that $m(N) = 0$. Let M be the union of all those points which belong to some partition Δ_k. Clearly, $m(M) = 0$ since M is denumerable. We claim that f is continuous off $N \cup M$. If f is not continuous at a point $c \notin N \cup M$, there is an $\varepsilon > 0$ and a sequence $(c_n)_{n=1}^{\infty}$ converging to c such that

$$| f(c_n) - f(c) | \geq \varepsilon \text{ all } n.$$

Since $c \notin M$, c is an interior point to exactly one interval of each partition Δ_k and we get

$$G_{\Delta_k}(c) - g_{\Delta_k}(c) \geq \varepsilon$$

and in the limit

$$G(c) - g(c) \geq \varepsilon.$$

But then $c \in N$ which is a contradiction.

Conversely, suppose the set of discontinuity points of f is a Lebesgue null set and let $(\Delta_k)_{k=1}^{\infty}$ is an arbitrary sequence of partitions of $[a,b]$ such that $\Delta_k \prec \Delta_{k+1}$ and $\text{mesh}(\Delta_k) \to 0$ as $k \to \infty$. By assumption,

$$\lim_{k\to\infty} G_{\Delta_k}(x) = \lim_{k\to\infty} g_{\Delta_k}(x) = f(x)$$

at each point x of continuity of f. Therefore f is Lebesgue measurable and dominated convergence yields

$$\lim_{k\to\infty}\int_{[a,b]} G_{\Delta_k}dm = \int_{[a,b]} f dm$$

and

$$\lim_{k\to\infty}\int_{[a,b]} g_{\Delta_k}dm = \int_{[a,b]} f dm.$$

Thus f is Riemann integrable and

$$\int_a^b f(x)dx = \int_{[a,b]} f dm.$$

In the following we sometimes write

$$\int_A f(x)dx \qquad (A \in \mathcal{R}^-)$$

instead of

$$\int_A f dm \qquad (A \in \mathcal{R}^-).$$

In a similar way we often prefer to write

$$\int_A f(x)dx \qquad (A \in \mathcal{R}_n^-)$$

instead of

$$\int_A f dm_n \qquad (A \in \mathcal{R}_n^-).$$

Furthermore, $\int_a^b f dm$ means $\int_{[a,b]} f dm$. Here, however, a warning is motivated. It is simple to find a real-valued function f on $[0,\infty[$, which is bounded on each bounded subinterval of $[0,\infty[$, such that the generalized Riemann integral

$$\int_0^\infty f(x)dx$$

is convergent, that is

$$\lim_{b\to\infty}\int_0^b f(x)dx$$

exists and the limit is a real number, while the Riemann integral

$$\int_0^\infty \mid f(x) \mid dx$$

is divergent (take e.g. $f(x) = \frac{\sin x}{x}$). In this case the function f does not belong to $\mathcal{L}^1$ with respect to Lebesgue measure on $[0, \infty[$ since

$$\int_{[0,\infty[} \mid f \mid dm = \lim_{b \to \infty} \int_0^b \mid f(x) \mid dx = \infty.$$

Exercises

1. Let $f_n : [0,1] \to [0,1]$, $n \in \mathbf{N}$, be a sequence of Riemann integrable functions such that

$$\lim_{n\to\infty} f_n(x) \text{ exists } = f(x) \text{ all } x \in [0,1]\,.$$

Show by giving an example that f need not be Riemann integrable.

2. Suppose $f_n(x) = n^2 \mid x \mid e^{-n|x|}$, $x \in \mathbf{R}$, $n \in \mathbf{N}_+$. Compute $\lim_{n\to\infty} f_n$ and $\lim_{n\to\infty} \int_{\mathbf{R}} f_n dm$.

3. Compute the following limits and justify the calculations:

a)

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(e^x)}{1+nx^2} dx.$$

b)

$$\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^{-n} \cos x dx.$$

c)

$$\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^{n} e^{-2x} dx.$$

d)

$$\lim_{n\to\infty} \int_0^\infty (1+\frac{x}{n})^{n} \exp(-(1+\frac{x}{n})^n) dx.$$

e)

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^{-n} e^{\frac{x}{2}} dx.$$

f)

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n \frac{1+nx}{n+x}\cos x dx$$

g)

$$\lim_{n\to\infty}\int_0^\infty (1+\frac{x}{n})^{n^2} e^{-nx} dx.$$

4. Let $(r_n)_{n=1}^\infty$ be an enumeration of $\mathbf{Q}$ and define

$$f(x)=\Sigma_{n=1}^\infty 2^{-n}\varphi(x-r_n)$$

where $\varphi(x)=x^{-\frac{1}{2}}$ if $0<x<1$ and $\varphi(x)=0$ if $x\leq 0$ or $x\geq 1$. Show that

a)

$$\int_{-\infty}^\infty f(x)dx=2.$$

b)

$$\int_a^b f^2(x)dx=\infty \text{ if } a<b.$$

c)

$$f<\infty \text{ a.s. } [m].$$

d)

$$\sup_{a<x<b} f(x)=+\infty \text{ if } a<b.$$

5. Suppose

$$f(t)=\int_0^\infty e^{-tx}\frac{\ln(1+x)}{1+x}dx,\ t>0.$$

a) Show that $\int_0^\infty f(t)dt<\infty$.
b) Show that f is infinitely many times differentiable.

2.4. Expectation

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\xi : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ a random variable. Recall that the probability law μ of ξ is given by the image measure P_ξ. By definition,

$$\int_S \chi_B d\mu = \int_\Omega \chi_B(\xi) dP$$

for every $B \in \mathcal{S}$, and, hence

$$\int_S \varphi d\mu = \int_\Omega \varphi(\xi) dP$$

for each simple $\mathcal{S}$-measurable function φ on S (we sometimes write $f \circ g = f(g)$). By monotone convergence, we get

$$\int_S f d\mu = \int_\Omega f(\xi) dP$$

for every measurable $f : S \to [0, \infty]$. Thus if $f : S \to \mathbf{R}$ is measurable, $f \in L^1(\mu)$ if and only if $f(\xi) \in L^1(P)$ and in this case

$$\int_S f d\mu = \int_\Omega f(\xi) dP.$$

In the special case when ξ is real-valued and $\xi \in L^1(P)$,

$$\int_{\mathbf{R}} x d\mu(x) = \int_\Omega \xi dP.$$

The integral in the right-hand side is called the expectation of ξ and is denoted by $E[\xi]$.