CHAPTER 4

MODES OF CONVERGENCE

Introduction

In this chapter we will treat a variety of different sorts of convergence notions in measure theory. So called L^2-convergence is of particular importance.

4.1. Convergence in Measure, in $L^1(\mu)$, and in $L^2(\mu)$

Let $(X, \mathcal{M}, \mu)$ be a positive measure space and denote by $\mathcal{F}(X)$ the class of measurable functions $f : (X, \mathcal{M}) \to (\mathbf{R}, \mathcal{R})$. For any $f \in \mathcal{F}(X)$, set

$$\| f \|_1 = \int_X \mid f(x) \mid d\mu(x)$$

and

$$\| f \|_2 = \sqrt{\int_X f^2(x) d\mu(x)}.$$

The Cauchy-Schwarz inequality states that

$$\int_X \mid fg \mid d\mu \leq \| f \|_2 \| g \|_2 \text{ if } f, g \in \mathcal{F}(X).$$

To prove this, without loss of generality, it can be assumed that

$$0 < \| f \|_2 < \infty \text{ and } 0 < \| g \|_2 < \infty.$$

We now use the inequality

$$\alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2),\ \alpha, \beta \in \mathbf{R}$$

to obtain

$$\int_X \frac{|f|}{\|f\|_2} \frac{|g|}{\|g\|_2} d\mu \leq \int \frac{1}{2}\left(\frac{f^2}{\|f\|_2^2} + \frac{g^2}{\|g\|_2^2}\right) d\mu = 1$$

and the Cauchy-Schwarz inequality is immediate.

If not otherwise stated, in this section p is a number equal to 1 or 2. If it is important to emphasize the underlying measure $\| f \|_p$ is written $\| f \|_{p,\mu}$.

We now define

$$\mathcal{L}^p(\mu) = \{f \in \mathcal{F}(X);\ \ \| f \|_p < \infty\}.$$

The special case $p = 1$ has been introduced earlier. We claim that the following so called triangle inequality holds, viz.

$$\| f + g \|_p \leq \| f \|_p + \| g \|_p \text{ if } f, g \in \mathcal{L}^p(\mu).$$

The case $p = 1$, follows by μ-integration of the relation

$$| f + g | \leq | f | + | g | .$$

To prove the case $p = 2$, we use the Cauchy-Schwarz inequality and have

$$\| f + g \|_2^2 \leq \| | f | + | g | \|_2^2$$

$$= \| f \|_2^2 + 2 \int_X | fg | \, d\mu + \| g \|_2^2$$

$$\leq \| f \|_2^2 + 2 \| f \|_2 \| g \|_2 + \| g \|_2^2 = (\| f \|_2 + \| g \|_2)^2$$

and the triangle inequality is immediate.

Suppose $f, g \in \mathcal{L}^p(\mu)$. The functions f and g are equal almost everywhere with respect to μ if $\{f \neq g\} \in \mathcal{Z}_\mu$. This is easily seen to be an equivalence relation and the set of all equivalence classes is denoted by $L^p(\mu)$. Below we consider the elements of $L^p(\mu)$ as members of $\mathcal{L}^p(\mu)$ and two members of $L^p(\mu)$ are identified if they are equal a.e. $[\mu]$. From this convention it is straight-forward to define $f + g$ and αf for all $f, g \in L^p(\mu)$ and $\alpha \in \mathbf{R}$ and the function $d^{(p)}(f, g) = \| f - g \|_p$ is a metric on $L^p(\mu)$. Convergence in the metric space $L^p(\mu) = (L^p(\mu), d^{(p)})$ is called convergence in $L^p(\mu)$. A sequence $(f_k)_{k=1}^\infty$ in $\mathcal{F}(X)$ converges in measure to a function $f \in \mathcal{F}(X)$ if

$$\lim_{k\to\infty} \mu(| f_k - f | > \varepsilon) = 0 \text{ all } \varepsilon > 0.$$

If the sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{F}(X)$ converges in measure to a function $f \in \mathcal{F}(X)$ as well as to a function $g \in \mathcal{F}(X)$, then $f = g$ a.e. $[\mu]$ since

$$\{\mid f - g \mid > \varepsilon\} \subseteq \left\{\mid f - f_k \mid > \frac{\varepsilon}{2}\right\} \cup \left\{\mid f_k - g \mid > \frac{\varepsilon}{2}\right\}$$

and

$$\mu(\mid f - g \mid > \varepsilon) \leq \mu(\mid f - f_k \mid > \frac{\varepsilon}{2}) + \mu(\mid f_k - g \mid > \frac{\varepsilon}{2})$$

for every $\varepsilon > 0$ and positive integer k. A sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{F}(X)$ is said to be Cauchy in measure if for every $\varepsilon > 0$,

$$\mu(\mid f_k - f_n \mid > \varepsilon) \to 0 \text{ as } k, n \to \infty.$$

By the Markov inequality, a Cauchy sequence in $L^p(\mu)$ is Cauchy in measure.

Example 4.1.1. (a) If $f_k = \sqrt{k}\chi_{[0,\frac{1}{k}]}$, $k \in \mathbf{N}_+$, then

$$\| f_k \|_{2,m} = 1 \text{ and } \| f_k \|_{1,m} = \frac{1}{\sqrt{k}}.$$

Thus $f_k \to 0$ in $L^1(m)$ as $k \to \infty$ but $f_k \nrightarrow 0$ in $L^2(m)$ as $k \to \infty$.

(b) $L^1(m) \nsubseteq L^2(m)$ since

$$\chi_{[1,\infty[}(x)\frac{1}{\mid x \mid} \in L^2(m) \setminus L^1(m)$$

and $L^2(m) \nsubseteq L^1(m)$ since

$$\chi_{]0,1]}(x)\frac{1}{\sqrt{\mid x \mid}} \in L^1(m) \setminus L^2(m).$$

Theorem 4.1.1. *Suppose $p = 1$ or 2.*

(a) *Convergence in $L^p(\mu)$ implies convergence in measure.*

(b) *If* $\mu(X) < \infty$, *then* $L^2(\mu) \subseteq L^1(\mu)$ *and convergence in* $L^2(\mu)$ *implies convergence in* $L^1(\mu)$.

Proof. (a) Suppose the sequence $(f_n)_{n=1}^{\infty}$ converges to f in $L^p(\mu)$ and let $\varepsilon > 0$. Then, by the Markov inequality,

$$\mu(\mid f_n - f \mid \geq \varepsilon) \leq \frac{1}{\varepsilon^p} \int_X \mid f_n - f \mid^p d\mu = \frac{1}{\varepsilon^p} \parallel f_n - f \parallel_p^p$$

and (a) follows at once.

(b) The Cauchy-Schwarz inequality gives for any $f \in \mathcal{F}(X)$,

$$(\int_X \mid f \mid \cdot 1 d\mu)^2 \leq \int_X f^2 d\mu \int_X 1 d\mu$$

or

$$\parallel f \parallel_1 \leq \parallel f \parallel_2 \sqrt{\mu(X)}$$

and Part (b) is immediate.

Theorem 4.1.2. *Suppose* $f_n \in \mathcal{F}(X)$, $n \in \mathbf{N}_+$.

(a) *If* $(f_n)_{n=1}^{\infty}$ *is Cauchy in measure, there is a measurable function* $f : X \to \mathbf{R}$ *such that* $f_n \to f$ *in measure as* $n \to \infty$ *and a strictly increasing sequence of positive integers* $(n_j)_{j=1}^{\infty}$ *such that* $f_{n_j} \to f$ *a.e.* $[\mu]$ *as* $j \to \infty$.

(b) *If* μ *is a finite positive measure and* $f_n \to f \in \mathcal{F}(X)$ *a.e.* $[\mu]$ *as* $n \to \infty$, *then* $f_n \to f$ *in measure.*

(c) **(Egoroff's Theorem)** *If* μ *is a finite positive measure and* $f_n \to f \in \mathcal{F}(X)$ *a.e.* $[\mu]$ *as* $n \to \infty$, *then for every* $\varepsilon > 0$ *there exists* $E \in \mathcal{M}$ *such that* $\mu(E) < \varepsilon$ and

$$\sup_{\substack{k \geq n \\ x \in E^c}} \mid f_k(x) - f(x) \mid \to 0 \text{ as } n \to \infty.$$

PROOF. (a) For each positive integer j, there is a positive integer n_j such that

$$\mu(\mid f_k - f_l \mid > 2^{-j}) < 2^{-j}, \text{ all } k, l \geq n_j.$$

There is no loss of generality to assume that $n_1 < n_2 < ...$. Set

$$E_j = \left\{ \mid f_{n_j} - f_{n_{j+1}} \mid > 2^{-j} \right\}$$

and

$$F_k = \cup_{j=k}^{\infty} E_j.$$

If $x \in F_k^c$ and $i \geq j \geq k$

$$\mid f_{n_i}(x) - f_{n_j}(x) \mid \leq \sum_{j \leq l < i} \mid f_{n_{l+1}}(x) - f_{n_l}(x) \mid$$

$$\leq \sum_{j \leq l < i} 2^{-l} < 2^{-j+1}$$

and we conclude that $(f_{n_j}(x))_{j=1}^{\infty}$ is a Cauchy sequence for every $x \in F_k^c$. Let $G = \cup_{k=1}^{\infty} F_k^c$ and note that for every fixed positive integer k,

$$\mu(G^c) \leq \mu(F_k) < \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}.$$

Thus G^c is a μ-null set. We now define $f(x) = \lim_{j\to\infty} f_{n_j}(x)$ if $x \in G$ and $f(x) = 0$ if $x \notin G$.

We next prove that the sequence $(f_n)_{n=1}^{\infty}$ converges to f in measure. If $x \in F_k^c$ and $j \geq k$ we get

$$\mid f(x) - f_{n_j}(x) \mid \leq 2^{-j+1}.$$

Thus, if $j \geq k$

$$\mu(\mid f - f_{n_j} \mid > 2^{-j+1}) \leq \mu(F_k) < 2^{-k+1}.$$

Since

$$\mu(\mid f_n - f \mid > \varepsilon) \leq \mu(\mid f_n - f_{n_j} \mid > \frac{\varepsilon}{2}) + \mu(\mid f_{n_j} - f \mid > \frac{\varepsilon}{2})$$

if $\varepsilon > 0$, Part (a) follows at once.

(b) For each $\varepsilon > 0$,

$$\mu(\mid f_n - f \mid > \varepsilon) = \int_X \chi_{]\varepsilon,\infty[}(\mid f_n - f \mid) d\mu$$

and Part (c) follows from the Lebesgue Dominated Convergence Theorem.

(c) Set for fixed $k, n \in \mathbf{N}_+$,

$$E_{kn} = \cup_{j=n}^{\infty} \left\{ \mid f_j - f \mid > \frac{1}{k} \right\}.$$

We have

$$\cap_{n=1}^{\infty} E_{kn} \in Z_\mu$$

and since μ is a finite measure

$$\mu(E_{kn}) \to 0 \text{ as } n \to \infty.$$

Given $\varepsilon > 0$ pick $n_k \in \mathbf{N}_+$ such that $\mu(E_{kn_k}) < \varepsilon 2^{-k}$. Then, if $E = \cup_{k=1}^{\infty} E_{kn_k}$, $\mu(E) < \varepsilon$. Moreover, if $x \notin E$ and $j \geq n_k$

$$\mid f_j(x) - f(x) \mid \leq \frac{1}{k}.$$

The theorem is proved.

Corollary 4.1.1. *The spaces $L^1(\mu)$ and $L^2(\mu)$ are complete.*

PROOF. Suppose $p = 1$ or 2 and let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\mu)$. We know from the previous theorem that there exists a subsequence $(f_{n_j})_{j=1}^{\infty}$ which converges pointwise to a function $f \in \mathcal{F}(X)$ a.e. $[\mu]$. Thus, by Fatou's Lemma,

$$\int_X \mid f - f_k \mid^p d\mu \leq \liminf_{j \to \infty} \int_X \mid f_{n_j} - f_k \mid^p d\mu$$

and it follows that $f - f_k \in L^p(\mu)$ and, hence $f = (f - f_k) + f_k \in L^p(\mu)$. Moreover, we have that $\| f - f_k \|_p \to 0$ as $k \to \infty$. This concludes the proof of the theorem.

Corollary 4.1.2. *Suppose $\xi_n \in N(0, \sigma_n^2)$, $n \in \mathbf{N}_+$, and $\xi_n \to \xi$ in $L^2(P)$ as $n \to \infty$. Then ξ is a centred Gaussian random variable.*

PROOF. We have that $\| \xi_n \|_2 = \sqrt{E\left[\xi_n^2\right]} = \sigma_n$ and $\| \xi_n \|_2 \to \| \xi \|_2 =_{def} \sigma$ as $n \to \infty$.

Suppose f is a bounded continuous function on $\mathbf{R}$. Then, by dominated convergence,

$$E\left[f(\xi_n)\right] = \int_{\mathbf{R}} f(\sigma_n x) d\gamma_1(x) \to \int_{\mathbf{R}} f(\sigma x) d\gamma_1(x)$$

as $n \to \infty$. Moreover, there exists a subsequence $(\xi_{n_k})_{k=1}^{\infty}$ which converges to ξ a.s. Hence, by dominated convergence

$$E\left[f(\xi_{n_k})\right] \to E\left[f(\xi)\right]$$

as $k \to \infty$ and it follows that

$$E\left[f(\xi)\right] = \int_{\mathbf{R}} f(\sigma x) d\gamma_1(x).$$

By using Corollary 3.1.3 the theorem follows at once.

Theorem 4.1.3. *Suppose X is a standard space and μ a positive σ-finite Borel measure on X. Then the spaces $L^1(\mu)$ and $L^2(\mu)$ are separable.*

PROOF. Let $(E_k)_{k=1}^{\infty}$ be a denumerable collection of Borel sets with finite μ-measures and such that $E_k \subseteq E_{k+1}$ and $\cup_{k=1}^{\infty} E_k = X$. Set $\mu_k = \chi_{E_k}\mu$ and first suppose that the set D_k is at most denumerable and dense in $L^p(\mu_k)$ for every $k \in \mathbf{N}_+$. Without loss of generality it can be assumed that each member of D_k vanishes off E_k. By monotone convergence

$$\int_X f d\mu = \lim_{k\to\infty} \int_X f d\mu_k, \; f \geq 0 \text{ measurable,}$$

and it follows that the set $\cup_{k=1}^{\infty} D_k$ is at most denumerable and dense in $L^p(\mu)$.

From now on we can assume that μ is a finite positive measure. Let A be an at most denumerable dense subset of X and and suppose the subset $\{r_n;\ n \in \mathbf{N}_+,\}$ of $\,]0,\infty[\,$ is dense in $]0,\infty[$. Furthermore, denote by $\mathcal{U}$ the class of all open sets which are finite unions of open balls of the type $B(a, r_n)$, $a \in A$, $n \in \mathbf{N}_+$. If U is any open subset of X

$$U = \cup [V : V \subseteq U \text{ and } V \in \mathcal{U}]$$

and, hence, by the Ulam Theorem

$$\mu(U) = \sup \{\mu(V);\ V \in \mathcal{U} \text{ and } V \subseteq U\}.$$

Let $\mathcal{K}$ be the class of all functions which are finite sums of functions of the type $\kappa\chi_U$, where κ is a positive rational number and $U \in \mathcal{U}$. It follows that $\mathcal{K}$ is at most denumerable.

Suppose $\varepsilon > 0$ and that $f \in L^p(\mu)$ is non-negative. There exists a sequence of simple measurable functions $(\varphi_i)_{i=1}^\infty$ such that

$$0 \leq \varphi_i \uparrow f \text{ a.e. } [\mu].$$

Since $| f - \varphi_i |^p \leq f^p$, the Lebesgue Dominated Convergence Theorem shows that $\| f - \varphi_k \|_p < \frac{\varepsilon}{2}$ for an appropriate k. Let $\alpha_1, ..., \alpha_l$ be the distinct positive values of φ_k and set

$$C = 1 + \Sigma_{k=1}^l \alpha_k.$$

Now for each fixed $j \in \{1, ..., l\}$ we use Theorem 3.1.3 to get an open $U_j \supseteq \varphi_k^{-1}(\{\alpha_j\})$ such that $\| \chi_{U_j} - \chi_{\varphi_k^{-1}(\{\alpha_j\})} \|_p < \frac{\varepsilon}{4C}$ and from the above we get a $V_j \in \mathcal{U}$ such that $V_j \subseteq U_j$ and $\| \chi_{U_j} - \chi_{V_j} \|_p < \frac{\varepsilon}{4C}$. Thus

$$\| \chi_{V_j} - \chi_{\varphi_k^{-1}(\{\alpha_j\})} \|_p < \frac{\varepsilon}{2C}$$

and

$$\| f - \Sigma_{k=1}^l \alpha_j \chi_{V_j} \|_p < \varepsilon$$

Now it is simple to find a $\psi \in \mathcal{K}$ such that $\| f - \psi \|_p < \varepsilon$. From this we deduce that the set

$$\mathcal{K} - \mathcal{K} = \{g - h;\ g, h \in \mathcal{K}\}$$

is at most denumerable and dense in $L^p(\mu)$.

The set of all real-valued and infinitely many times differentiable functions defined on $\mathbf{R}^n$ is denoted by $C^{(\infty)}(\mathbf{R}^n)$ and

$$C_c^{(\infty)}(\mathbf{R}^n) = \left\{ f \in C^{(\infty)}(\mathbf{R}^n);\ \operatorname{supp} f \text{ compact} \right\}.$$

Recall that the support $\operatorname{supp} f$ of a real-valued continuous function f defined on $\mathbf{R}^n$ is the closure of the set of all x where $f(x) \neq 0$. If

$$f(x) = \prod_{k=1}^{n} \left\{ \varphi(1 + x_k)\varphi(1 - x_k) \right\},\ x = (x_1, ..., x_n) \in \mathbf{R}^n$$

where $\varphi(t) = \exp(-t^{-1})$, if $t > 0$, and $\varphi(t) = 0$, if $t \leq 0$, then $f \in C_c^{\infty}(\mathbf{R}^n)$.

The proof of the previous theorem also gives Part (a) of the following

Theorem 4.1.4. *Suppose μ is a positive Borel measure in $\mathbf{R}^n$ such that $\mu(K) < \infty$ for every compact subset K of $\mathbf{R}^n$. The following sets are dense in $L^1(\mu)$, and $L^2(\mu)$:*

(a) *the linear span of the functions*

$$\chi_I,\ I \text{ open bounded } n\text{-cell in } \mathbf{R}^n,$$

(b) $C_c^{(\infty)}(\mathbf{R}^n)$.

PROOF. a) The proof is almost the same as the proof of Theorem 4.1.3. First the E_k:s can be chosen to be open balls with their centres at the origin since each bounded set in $\mathbf{R}^n$ has finite μ-measure. Moreover, as in the proof of Theorem 4.1.3 we can assume that μ is a finite measure. Now let A be an at most denumerable dense subset of $\mathbf{R}^n$ and for each $a \in A$ let

$$R(a) = \{ r > 0;\ \mu(\{ x \in X;\ \mid x_k - a_k \mid = r \}) > 0 \text{ for some } k = 1, ..., n \}.$$

Then $\cup_{a \in A} R(a)$ is at most denumerable and there is a subset $\{ r_n;\ n \in \mathbf{N}_+ \}$ of $]0, \infty[\setminus \cup_{a \in A} R(a)$ which is dense in $]0, \infty[$. Finally, let $\mathcal{U}$ denote the class of all open sets which are finite unions of open balls of the type $B(a, r_n)$, $a \in A$, $n \in \mathbf{N}_+$, and proceed as in the proof of Theorem 4.1.3. The result follows by observing that the characteristic function of any member of $\mathcal{U}$ equals a finite sum of characteristic functions of open bounded n-cells a.e. $[\mu]$.

Part (b) in Theorem 4.1.4 follows from Part (a) and the following

Lemma 4.1.1. *Suppose $K \subseteq U \subseteq \mathbf{R}^n$, where K is compact and U is open. Then there exists a function $f \in C_c^{(\infty)}(\mathbf{R}^n)$* such that

$$K \prec f \prec U$$

that is

$$\chi_K \leq f \leq \chi_U \text{ and supp} f \subseteq U.$$

PROOF. Suppose $\rho \in C_c^\infty(\mathbf{R}^n)$ is non-negative, supp $\rho \subseteq B(0,1)$, and

$$\int_{\mathbf{R}^n} \rho dm_n = 1.$$

Moreover, let $\varepsilon > 0$ be fixed. For any $g \in L^1(v_n)$ we define

$$f_\varepsilon(x) = \varepsilon^{-n} \int_{\mathbf{R}^n} g(y)\rho(\varepsilon^{-1}(x-y))dy.$$

Since

$$\mid g \mid \max_{\mathbf{R}^n} \mid \frac{\partial^{k_1+...+k_n}\rho}{\partial x_1^{k_1}...\partial x_n^{k_n}} \mid \in L^1(v_n), \text{ all } k_1,...,k_n \in \mathbf{N}$$

the Lebesgue Dominated Convergent Theorem shows that $f_\varepsilon \in C^\infty(\mathbf{R}^n)$. Here $f_\varepsilon \in C_c^\infty(\mathbf{R}^n)$ if g vanishes off a bounded subset of $\mathbf{R}^n$. In fact,

$$\text{supp } f_\varepsilon \subseteq (\text{supp } g)_\varepsilon.$$

Now choose a positive number $\varepsilon \leq \frac{1}{2}d(K, U^c)$ and define $g = \chi_{K_\varepsilon}$. Since

$$f_\varepsilon(x) = \int_{\mathbf{R}^n} g(x - \varepsilon y)\rho(y)dy$$

we also have that $f_\varepsilon(x) = 1$ if $x \in K$. The lemma is proved.

Example 4.1.2. Suppose $f \in L^1(m_n)$ and let $g : \mathbf{R}^n \to \mathbf{R}$ be a bounded Lebesgue measurable function. Set

$$h(x) = \int_{\mathbf{R}^n} f(x-y)g(y)dy, \ x \in \mathbf{R}^n.$$

We claim that h is continuous.

To see this first note that

$$h(x+\Delta x) - h(x) = \int_{\mathbf{R}^n} (f(x+\Delta x - y) - f(x-y))g(y)dy$$

and

$$\mid h(x+\Delta x) - h(x) \mid \leq K \int_{\mathbf{R}^n} \mid f(x+\Delta x - y) - f(x-y) \mid dy$$

$$= K \int_{\mathbf{R}^n} \mid f(\Delta x + y) - f(y) \mid dy$$

if $\mid g(x) \mid \leq K$ for every $x \in \mathbf{R}^n$. Now first choose $\varepsilon > 0$ and then $\varphi \in C_c(\mathbf{R}^n)$ such that

$$\parallel f - \varphi \parallel_1 < \varepsilon.$$

Using the triangle inequality, we get

$$\mid h(x+\Delta x) - h(x) \mid \leq K(2 \parallel f - \varphi \parallel_1 + \int_{\mathbf{R}^n} \mid \varphi(\Delta x + y) - \varphi(y) \mid dy)$$

$$\leq K(2\varepsilon + \int_{\mathbf{R}^n} \mid \varphi(\Delta x + y) - \varphi(y) \mid dy)$$

where the right hand side is smaller than $3K\varepsilon$ if $\mid \Delta x \mid$ is sufficiently small. This proves that h is continuous.

Example 4.1.3. Suppose $A \in \mathcal{R}_n^-$ and $m_n(A) > 0$. We claim that the set

$$A - A = \{x - x; x \in A\}$$

contains a neighbourhood of the origin.

To show this there is no loss of generality to assume that $m_n(A) < \infty$. Set

$$f(x) = m_n(A \cap (A + x)), \ x \in \mathbf{R}^n.$$

Note that

$$f(x) = \int_{\mathbf{R}^n} \chi_A(y)\chi_A(y-x)dy$$

and Example 4.1.2 proves that f is continuous. Since $f(0) > 0$ there exists a $\delta > 0$ such that $f(x) > 0$ if $| x |< \delta$. In particular, $A \cap (A+x) \neq \phi$ if $| x |< \delta$, which proves that

$$B(0, \delta) \subseteq A - A.$$

The following three examples are based on the Axiom of Choice.

Example 4.1.4. Let NL be the non-Lebesgue measurable set constructed in Section 1.3. Furthermore, assume $A \subseteq \mathbf{R}$ is Lebesgue measurable and $A \subseteq NL$. We claim that $m(A) = 0$. If not, there exists a $\delta > 0$ such that $B(0, \delta) \subseteq A - A \subseteq NL - NL$. If $0 < r < \delta$ and $r \in \mathbf{Q}$, there exist $a, b \in NL$ such that

$$a = b + r.$$

But then $a \neq b$ and at the same time a and b belong to the same equivalence class, which is a contradiction. Accordingly from this, $m(A) = 0$.

Example 4.1.5. Suppose $A \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$ is Lebesgue measurable and $m(A) > 0$. We claim there exists a non-Lebesgue measurable subset of A. To see this note that

$$A = \cup_{i=1}^{\infty}((r_i + NL) \cap A)$$

where $(r_i)_{i=1}^{\infty}$ is an enumeration of the rational numbers in the interval $[-1, 1]$. If each set $(r_i + NL) \cap A$, is Lebesgue measurable

$$m(A) = \Sigma_{i=1}^{\infty} m((r_i + NL) \cap A)$$

and we conclude that $m((r_i + NL) \cap A) > 0$ for an appropriate i. But then $m(NL \cap (A - r_i)) > 0$ and $NL \cap (A - r_i) \subseteq NL$, which contradicts Example 4.1.4. Hence $(r_i + NL) \cap A$ is non-Lebesgue measurable for an appropriate i.

If A is a Lebesgue measurable subset of the real line of positive Lebesgue measure, we conclude that A contains a non-Lebesgue measurable subset.

Example 4.1.6. Set $I = [0,1]$. We claim there exist a continuous function $f : I \to I$ and a Lebesgue measurable set $L \subseteq I$ such that $f^{-1}(L)$ is not Lebesgue measurable.

First recall from Section 3.3 the construction of the Cantor set C and the Cantor function G. First $C_0 = [0,1]$. Then trisect C_0 and remove the middle interval $\left]\frac{1}{3}, \frac{2}{3}\right[$ to obtain $C_1 = C_0 \setminus \left]\frac{1}{3}, \frac{2}{3}\right[= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. At the second stage subdivide each of the closed intervals of C_1 into thirds and remove from each one the middle open thirds. Then $C_2 = C_1 \setminus (\left]\frac{1}{9}, \frac{2}{9}\right[\cup \left]\frac{7}{9}, \frac{8}{9}\right[)$. We repeat the process and what is left from C_{n-1} is C_n. The set $[0,1] \setminus C_n$ is the union of $2^n - 1$ intervals numbered I_k^n, $k = 1, ..., 2^n - 1$, where the interval I_k^n is situated to the left of the interval I_l^n if $k < l$. The Cantor set $C = \cap_{n=1}^{\infty} C_n$.

Suppose n is fixed and let $G_n : [0,1] \to [0,1]$ be the unique the monotone increasing continuous function, which satisfies $G_n(0) = 0, G_n(1) = 1, G_n(x) = k2^{-n}$ for $x \in I_k^n$ and which is linear on each interval of $C_{n.}$ It is clear that $G_n = G_{n+1}$ on each interval I_k^n, $k = 1, ..., 2^n - 1$. The Cantor function is defined by the limit $G(x) = \lim_{n\to\infty} G_n(x)$, $0 \leq x \leq 1$.

Now define

$$h(x) = \frac{1}{2}(x + G(x)),\ x \in I$$

where G is the Cantor function. Since $h : I \to I$ is a strictly increasing and continuous bijection, the inverse function $f = h^{-1}$ is a continuous bijection from I onto I. Set

$$A = h(I \setminus C)$$

and

$$B = h(C).$$

Recall from the definition of G that G is constant on each removed interval I_k^n and that h takes each removed interval onto an interval of half its length. Thus $m(A) = \frac{1}{2}$ and $m(B) = 1 - m(A) = \frac{1}{2}$.

By the previous example there exists a non-Lebesgue measurable subset M of B. Put $L = h^{-1}(M)$. The set L is Lebesgue measurable since $L \subseteq C$ and C is a Lebesgue null set. However, the set $M = f^{-1}(L)$ is not Lebesgue measurable.

Exercises

1. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space and suppose $\varphi(t) = \min(t, 1)$, $t \geq 0$. Prove that $f_n \to f$ in measure if and only if $\varphi(| f_n - f |) \to 0$ in $L^1(\mu)$.

2. Let $\mu = m_{|[0,1]}$. Find measurable functions $f_n : [0, 1] \to [0, 1]$, $n \in \mathbf{N}_+$, such that $f_n \to 0$ in $L^2(\mu)$ as $n \to \infty$,

$$\liminf_{n\to\infty} f_n(x) = 0 \text{ all } x \in [0, 1]$$

and

$$\limsup_{n\to\infty} f_n(x) = 1 \text{ all } x \in [0, 1] .$$

3. If $f \in \mathcal{F}(X)$ set

$$\| f \|_0 = \inf \{\alpha \in [0, \infty] ;\ \mu(| f | > \alpha) \leq \alpha\} .$$

Let

$$L^0(\mu) = \{f \in \mathcal{F}(X);\ \ \| f \|_0 < \infty\}$$

and identify functions in $L^0(\mu)$ which agree a.e. $[\mu]$.

(a) Prove that $d^{(0)} = \| f - g \|_0$ is a metric on $L^0(\mu)$ and that the corresponding metric space is complete.

(b) Show that $\mathcal{F}(X) = L^0(\mu)$ if μ is a finite positive measure.

4. Suppose $L^p(X, \mathcal{M}, \mu)$ is separable, where $p = 1$ or 2. Show that $L^p(X, \mathcal{M}^-, \bar{\mu})$ is separable.

5. Suppose g is a real-valued, Lebesgue measurable, and bounded function of period one. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)g(nx)dx = \int_{-\infty}^{\infty} f(x)dx \int_0^1 g(x)dx$$

for every $f \in L^1(m)$.

6. Let $h_n(t) = 2 + \sin nt$, $0 \leq t \leq 1$, and $n \in \mathbf{N}_+$. Find real constants α and β such that

$$\lim_{n\to\infty} \int_0^1 f(t)h_n(t)dt = \alpha \int_0^1 f(t)dt$$

and

$$\lim_{n\to\infty} \int_0^1 \frac{f(t)}{h_n(t)}dt = \beta \int_0^1 f(t)dt$$

for every real-valued Lebesgue integrable function f on $[0,1]$.

7. If $k = (k_1, ..., k_n) \in \mathbf{N}_+^n$, set $e_k(x) = \Pi_{i=1}^n \sin k_i x_i$, $x = (x_1, ..., x_n) \in \mathbf{R}^n$, and $\mid k \mid = (\Sigma_{i=1}^n k_i^2)^{\frac{1}{2}}$. Prove that

$$\lim_{|k|\to\infty} \int_{\mathbf{R}^n} f e_k dm_n = 0$$

for every $f \in L^1(m_n)$.

8. Suppose $f \in L^1(m_n)$, where m_n denotes Lebesgue measure on $\mathbf{R}^n$. Compute the following limit and justify the calculations:

$$\lim_{|h|\to\infty} \int_{\mathbf{R}^n} \mid f(x+h) - f(x) \mid dx.$$

4.2 Orthogonality

Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space. If $f, g \in L^2(\mu)$, let

$$\langle f, g \rangle =_{def} \int_X fg d\mu$$

be the so called scalar product of f and g. The Cauchy-Schwarz inequality

$$\mid \langle f, g \rangle \mid \leq \parallel f \parallel_2 \parallel g \parallel_2$$

shows that the map $f \to \langle f, g \rangle$ of $L^2(\mu)$ into $\mathbf{R}$ is continuous. Observe that

$$\| f + g \|_2^2 = \| f \|_2^2 + 2\langle f, g \rangle + \| g \|_2^2$$

and from this we get the so called Parallelogram Law

$$\| f + g \|_2^2 + \| f - g \|_2^2 = 2(\| f \|_2^2 + \| g \|_2^2).$$

We will say that f and g are orthogonal (abbr. $f \perp g$) if $\langle f, g \rangle = 0$. Note that

$$\| f + g \|_2^2 = \| f \|_2^2 + \| g \|_2^2 \text{ if and only if } f \perp g.$$

Since $f \perp g$ implies $g \perp f$, the relation $\perp$ is symmetric. Moreover, if $f \perp h$ and $g \perp h$ then $(\alpha f + \beta g) \perp h$ for all $\alpha, \beta \in \mathbf{R}$. Thus $h^\perp =_{def} \{ f \in L^2(\mu);\ f \perp h \}$ is a subspace of $L^2(\mu)$, which is closed since the map $f \to \langle f, h \rangle$, $f \in L^2(\mu)$ is continuous. If M is a subspace of $L^2(\mu)$, the set

$$M^\perp =_{def} \cap_{h \in M} h^\perp$$

is a closed subspace of $L^2(\mu)$. The function $f = 0$ if and only if $f \perp f$.

If M is a subspace of $L^2(\mu)$ and $f \in L^2(\mu)$ there exists at most one point $g \in M$ such that $f - g \in M^\perp$. To see this, let $g_0, g_1 \in M$ be such that $f - g_k \in M^\perp$, $k = 0, 1$. Then $g_1 - g_0 = (f - g_0) - (f - g_1) \in M^\perp$ and hence $g_1 - g_0 \perp g_1 - g_0$ that is $g_0 = g_1$.

Theorem 4.2.1. *Let M be a closed subspace in $L^2(\mu)$ and suppose $f \in L^2(\mu)$. Then there exists a unique point $g \in M$ such that*

$$\| f - g \|_2 \leq \| f - h \|_2 \quad \text{all } h \in M.$$

Moreover,

$$f - g \in M^\perp.$$

The function g in Theorem 4.2.1 is called the projection of f on M and is denoted by $\mathrm{Proj}_M\ f$.

PROOF OF THEOREM 4.2.1. Set

$$d =_{def} d^{(2)}(f, M) = \inf_{g \in M} \| f - g \|_2 .$$

and let $(g_n)_{n=1}^{\infty}$ be a sequence in M such that

$$d = \lim_{n \to \infty} \| f - g_n \|_2 .$$

Then, by the Parallelogram Law

$$\| (f-g_k)+(f-g_n) \|_2^2 + \| (f-g_k)-(f-g_n) \|_2^2 = 2(\| f-g_k \|_2^2 + \| f-g_n \|_2^2)$$

that is

$$4 \| f - \frac{1}{2}(g_k + g_n) \|_2^2 + \| g_n - g_k \|_2^2 = 2(\| f - g_k \|_2^2 + \| f - g_n \|_2^2)$$

and, since $\frac{1}{2}(g_k + g_n) \in M$, we get

$$4d^2 + \| g_n - g_k \|_2^2 \leq 2(\| f - g_k \|_2^2 + \| f - g_n \|_2^2).$$

Here the right hand converges to $4d^2$ as k and n go to infinity and we conclude that $(g_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since $L^2(\mu)$ is complete and M closed there exists a $g \in M$ such that $g_n \to g$ as $n \to \infty$. Moreover,

$$d = \| f - g \|_2 .$$

We claim that $f - g \in M^{\perp}$. To prove this choose $h \in M$ and $\alpha > 0$ arbitrarily and use the inequality

$$\| (f - g) + \alpha h \|_2^2 \geq \| f - g \|_2^2$$

to obtain

$$\| f - g \|_2^2 + 2\alpha\langle f - g, h\rangle + \alpha^2 \| h \|_2^2 \geq \| f - g \|_2^2$$

and

$$2\langle f - g, h\rangle + \alpha \| h \|_2^2 \geq 0.$$

By letting $\alpha \to 0$, $\langle f - g, h\rangle \geq 0$ and replacing h by $-h$, $\langle f - g, h\rangle \leq 0$. Thus $f - g \in h^{\perp}$ and it follows that $f - g \in M^{\perp}$.

The uniqueness in Theorem 4.2.1 follows from the remark just before the formulation of Theorem 4.2.1. The theorem is proved.

A linear mapping $T : L^2(\mu) \to \mathbf{R}$ is called a linear functional on $L^2(\mu)$. If $h \in L^2(\mu)$, the map $h \to \langle f, h\rangle$ of $L^2(\mu)$ into $\mathbf{R}$ is a continuous linear functional on $L^2(\mu)$. It is a very important fact that every continuous linear functional on $L^2(\mu)$ is of this type.

Theorem 4.2.2. *Suppose T is a continuous linear functional on $L^2(\mu)$. Then there exists a unique $w \in L^2(\mu)$ such that*

$$Tf = \langle f, w \rangle \text{ all } f \in L^2(\mu).$$

PROOF. Uniqueness: If $w, w' \in L^2(\mu)$ and $\langle f, w \rangle = \langle f, w' \rangle$ for all $f \in L^2(\mu)$, then $\langle f, w - w' \rangle = 0$ for all $f \in L^2(\mu)$. By choosing $f = w - w'$ we get $f \perp f$ that is $w = w'$.

Existence: The set $M =_{def} T^{-1}(\{0\})$ is closed since T is continuous and M is a linear subspace of $L^2(\mu)$ since T is linear. If $M = L^2(\mu)$ we choose $w = 0$. Otherwise, pick a $g \in L^2(\mu) \setminus M$. Without loss of generality it can be assumed that $Tg = 1$ by eventually multiplying g by a scalar. The previous theorem gives us a vector $h \in M$ such that $u =_{def} g - h \in M^{\perp}$. Note that $0 < \| u \|_2^2 = \langle u, g - h \rangle = \langle u, g \rangle$.

To conclude the proof, let fixed $f \in L^2(\mu)$ be fixed, and use that $(Tf)g - f \in M$ to obtain

$$\langle (Tf)g - f, u \rangle = 0$$

or

$$(Tf)\langle g, u \rangle = \langle f, u \rangle.$$

By setting

$$w = \frac{1}{\| u \|_2^2} u$$

we are done.

$\downarrow\downarrow\downarrow$

4.3. The Haar Basis and Wiener Measure

In this section we will show the existence of Brownian motion with continuous paths as a consequence of the existence of linear measure λ in the unit interval. The so called Wiener measure is the probability law on $C\,[0,1]$ of real-valued Brownian motion in the time interval $[0,1]$. The Brownian motion process is named after the British botanist Robert Brown (1773-1858). It was suggested by Lous Bachelier in 1900 as a model of stock price fluctuations and later by Albert Einstein in 1905 as a model of the physical phenomenon Brownian motion. The existence of the mathematical Brownian motion process was first established by Norbert Wiener in the twenties. Wiener also proved that the model can be chosen such that the path $t \to W(t)$, $0 \leq t \leq 1$, is continuous a.s. Today Brownian motion is a very important concept in probability, financial mathematics, partial differential equations and in many other fields in pure and applied mathematics.

Suppose n is a non-negative integer and set $I_n = \{0, ..., n\}$. A sequence $(e_i)_{i \in I_n}$ in $L^2(\mu)$ is said to be orthonormal if $e_i \perp e_j$ for all $i \neq j$, $i, j \in I_n$ and $\parallel e_i \parallel = 1$ for each $i \in I_n$. If $(e_i)_{i \in I_n}$ is orthonormal and $f \in L^2(\mu)$,

$$f - \Sigma_{i \in I_n} \langle f, e_i \rangle e_i \perp e_j \text{ all } j \in I$$

and Theorem 4.2.1 shows that

$$\parallel f - \Sigma_{i \in I_n} \langle f, e_i \rangle e_i \parallel_2 \leq \parallel f - \Sigma_{i \in I_n} \alpha_i e_i \parallel_2 \text{ all real } \alpha_1, ..., \alpha_n.$$

Moreover

$$\parallel f \parallel_2^2 = \parallel f - \Sigma_{i \in I_n} \langle f, e_i \rangle e_i \parallel_2^2 + \parallel \Sigma_{i \in I_n} \langle f, e_i \rangle e_i \parallel_2^2$$

and we get

$$\Sigma_{i \in I_n} \langle f, e_i \rangle^2 \leq \parallel f \parallel_2^2 .$$

We say that $(e_n)_{n \in I_n}$ is an orthonormal basis in $L^2(\mu)$ if it is orthonormal and

$$f = \Sigma_{i \in I_n} \langle f, e_i \rangle e_i \text{ all } f \in L^2(\mu).$$

A sequence $(e_i)_{i=0}^{\infty}$ in $L^2(\mu)$ is said to be orthonormal if $(e_i)_{i=0}^{n}$ is orthonormal for each non-negative integer n. In this case, for each $f \in L^2(\mu)$,

$$\Sigma_{i=0}^{\infty} \langle f, e_i \rangle^2 \leq \parallel f \parallel_2^2$$

and the series

$$\Sigma_{i=0}^{\infty} \langle f, e_i \rangle e_i$$

converges since the sequence

$$(\Sigma_{i=0}^{n}\langle f, e_i\rangle e_i)_{n=0}^{\infty}$$

of partial sums is a Cauchy sequence in $L^2(\mu)$. We say that $(e_i)_{i=0}^{\infty}$ is an orthonormal basis in $L^2(\mu)$ if it is orthonormal and

$$f = \Sigma_{i=0}^{\infty}\langle f, e_i\rangle e_i \text{ for all } f \in L^2(\mu).$$

Theorem 4.3.1. *An orthonormal sequence $(e_i)_{i=0}^{\infty}$ in $L^2(\mu)$ is a basis of $L^2(\mu)$ if*

$$(\langle f, e_i\rangle = 0 \text{ all } i \in \mathbf{N}) \Rightarrow f = 0$$

Proof. Let $f \in L^2(\mu)$ and set

$$g = f - \Sigma_{i=0}^{\infty}\langle f, e_i\rangle e_i.$$

Then, for any $j \in \mathbf{N}$,

$$\langle g, e_j\rangle = \langle f - \Sigma_{i=0}^{\infty}\langle f, e_i\rangle e_i, e_j\rangle$$

$$= \langle f, e_j\rangle - \Sigma_{i=0}^{\infty}\langle f, e_i\rangle\langle e_i, e_j\rangle = \langle f, e_j\rangle - \langle f, e_j\rangle = 0.$$

Thus $g = 0$ or

$$f = \Sigma_{i=0}^{\infty}\langle f, e_i\rangle e_i.$$

The theorem is proved.

As an example of an application of Theorem 4.3.1 we next construct an orthonormal basis of $L^2(\lambda)$, where λ is linear measure in the unit interval. Set

$$H(t) = \chi_{[0,\frac{1}{2}[}(t) - \chi_{[\frac{1}{2},1]}(t),\ t \in \mathbf{R}$$

Moreover, define $h_{00}(t) = 1$, $0 \leq t \leq 1$, and for each $n \geq 1$ and $j = 1, ..., 2^{n-1}$,

$$h_{jn}(t) = 2^{\frac{n-1}{2}} H(2^{n-1}t - j + 1),\ 0 \leq t \leq 1.$$

Stated otherwise, we have for each $n \geq 1$ and $j = 1, ..., 2^{n-1}$

$$h_{jn}(t) = \begin{cases} 2^{\frac{n-1}{2}}, \frac{j-1}{2^{n-1}} \leq t < \frac{j-\frac{1}{2}}{2^{n-1}}, \\ \\ -2^{\frac{n-1}{2}}, \frac{j-\frac{1}{2}}{2^{n-1}} \leq t \leq \frac{j}{2^{n-1}}, \\ \\ 0, \text{ elsewhere in } [0,1]. \end{cases}$$

It is simple to show that the sequence $h_{00,}h_{jn}, j = 1, ..., 2^{n-1}, n \geq 1$, is orthonormal in $L^2(\lambda)$. We will prove that the same sequence constitute an orthonormal basis of $L^2(\lambda)$. Therefore, suppose $f \in L^2(\lambda)$ is orthogonal to each of the functions $h_{00,}h_{jn}, j = 1, ..., 2^{n-1}, n \geq 1$. Then for each $n \geq 1$ and $j = 1, ..., 2^{n-1}$

$$\int_{\frac{j-1}{2^{n-1}}}^{\frac{j-\frac{1}{2}}{2^{n-1}}} f d\lambda = \int_{\frac{j-\frac{1}{2}}{2^{n-1}}}^{\frac{j}{2^{n-1}}} f d\lambda$$

and, hence,

$$\int_{\frac{j-1}{2^{n-1}}}^{\frac{j}{2^{n-1}}} f d\lambda = \frac{1}{2^{n-1}} \int_0^1 f d\lambda = 0$$

since

$$\int_0^1 f d\lambda = \int_0^1 f h_{00} d\lambda = 0.$$

Thus

$$\int_{\frac{j}{2^{n-1}}}^{\frac{k}{2^{n-1}}} f d\lambda = 0, \ 1 \leq j \leq k \leq 2^{n-1}$$

and we conclude that

$$\int_0^1 1_{[a,b]} f d\lambda = \int_a^b f d\lambda = 0, \ 0 \leq a \leq b \leq 1.$$

Accordingly from this, $f = 0$ and we are done.

The above basis $(h_k)_{k=0}^{\infty} = (h_{00,}h_{11}, h_{12}, h_{22}, h_{13}, h_{23}, h_{33}, h_{43}, ...)$ of $L^2(\lambda)$ is called the Haar basis.

Let $0 \leq t \leq 1$ and define for fixed $k \in \mathbf{N}$

$$a_k(t) = \int_0^1 \chi_{[0,t]}(x) h_k(x) dx = \int_0^t h_k d\lambda$$

so that

$$\chi_{[0,t]} = \Sigma_{k=0}^{\infty} a_k(t) h_k \text{ in } L^2(\lambda).$$

Then, if $0 \leq s, t \leq 1$,

$$\min(s,t) = \int_0^1 \chi_{[0,s]}(x)\chi_{[0,t]}(x)dx = \langle \Sigma_{k=1}^{\infty} a_k(s) h_k, \chi_{[0,t]} \rangle$$

$$= \Sigma_{k=0}^{\infty} a_k(s) \langle h_k, \chi_{[0,t]} \rangle = \Sigma_{k=0}^{\infty} a_k(s) a_k(t).$$

Note that

$$t = \Sigma_{k=0}^{\infty} a_k^2(t).$$

If $(G_k)_{k=0}^{\infty}$ is a sequence of $N(0,1)$ distributed random variables based on a probability space $(\Omega, \mathcal{F}, P)$ the series

$$\Sigma_{k=0}^{\infty} a_k(t) G_k$$

converges in $L^2(P)$ and defines a Gaussian random variable which we denote by $W(t)$. From the above it follows that $(W(t))_{0 \leq t \leq 1}$ is a real-valued centred Gaussian stochastic process with the covariance

$$E\left[W(s)W(t)\right] = \min(s,t).$$

Such a process is called a real-valued Brownian motion in the time interval $[0,1]$.

Recall that

$$(h_{00,} h_{11}, h_{12}, h_{22}, h_{13}, h_{23}, h_{33}, h_{43}, ...) = (h_k)_{k=0}^{\infty}.$$

We define

$$(a_{00,} a_{11}, a_{12}, a_{22}, a_{13}, a_{23}, a_{33}, a_{43}, ...) = (a_k)_{k=0}^{\infty}$$

and

$$(G_{00,} G_{11}, G_{12}, G_{22}, G_{13}, G_{23}, G_{33}, G_{43}, ...) = (G_k)_{k=0}^{\infty}.$$

It is important to note that for fixed n,

$$a_{jn}(t) = \int_0^t \chi_{[0,t]}(x) h_{jn}(x) dx \neq 0 \text{ for at most one } j.$$

Set

$$U_0(t) = a_{00}(t) G_{00}$$

and

$$U_n(t) = \Sigma_{j=1}^{2^{n-1}} a_{jn}(t) G_{jn},\ n \in \mathbf{N}_+.$$

We know that

$$W(t) = \Sigma_{n=0}^{\infty} U_n(t) \text{ in } L^2(P)$$

for fixed t.

The space $C\,[0,1]$ will from now on be equipped with the metric

$$d(x,y) = \parallel x - y \parallel_\infty$$

where $\parallel x \parallel_\infty = \max_{0 \leq t \leq 1} \mid x(t) \mid$. Recall that every $x \in C\,[0,1]$ is uniformly continuous. From this, remembering that $\mathbf{R}$ is separable, it follows that the space $C\,[0,1]$ is separable. Since $\mathbf{R}$ is complete it is also simple to show that the metric space $C\,[0,1]$ is complete. Finally, if $x_n \in C\,[0,1]$, $n \in \mathbf{N}$, and

$$\Sigma_{n=0}^{\infty} \parallel x_n \parallel_\infty < \infty$$

the series

$$\Sigma_{n=0}^{\infty} x_n$$

converges since the partial sums

$$s_n = \Sigma_{k=0}^{n} x_k,\ k \in \mathbf{N}$$

forms a Cauchy sequence.

We now define

$$\Theta = \left\{ \omega \in \Omega; \Sigma_{n=0}^{\infty} \parallel U_n \parallel_\infty < \infty \right\}.$$

Here $\Theta \in \mathcal{F}$ since

$$\parallel U_n \parallel_\infty = \sup_{\substack{0 \leq t \leq 1 \\ t \in \mathbf{Q}}} \mid U_n(t) \mid$$

for each n. Next we prove that $\Omega \setminus \Theta$ is a null set.

To this end let $n \geq 1$ and note that

$$P\left[\parallel U_n \parallel_\infty > 2^{-\frac{n}{4}}\right] \leq P\left[\max_{1 \leq j \leq 2^{n-1}} (\parallel a_{jn} \parallel_\infty \mid G_{jn} \mid) > 2^{-\frac{n}{4}}\right].$$

But

$$\parallel a_{jn} \parallel_\infty = \frac{1}{2^{\frac{n+1}{2}}}$$

and, hence,

$$P\left[\| U_n \|_\infty > 2^{-\frac{n}{4}}\right] \leq 2^{n-1} P\left[| G_{00} | > 2^{\frac{n}{4}+\frac{1}{2}}\right].$$

Since

$$x \geq 1 \Rightarrow P\left[| G_{00} | \geq x\right] \leq 2\int_x^\infty y e^{-y^2/2} \frac{dy}{x\sqrt{2\pi}} \leq e^{-x^2/2}$$

we get

$$P\left[\| U_n \|_\infty > 2^{-\frac{n}{4}}\right] \leq 2^n e^{-2^{n/2}}$$

and conclude that

$$E\left[\sum_{n=0}^\infty 1_{\left[\| U_n \|_\infty > 2^{-\frac{n}{4}}\right]}\right] = \sum_{n=0}^\infty P\left[\| U_n \|_\infty > 2^{-\frac{n}{4}}\right] < \infty.$$

From this and the Beppo Levi Theorem (or the first Borel-Cantelli Lemma) $P[\Theta] = 1$.

The trajectory $t \to W(t,\omega)$, $0 \leq t \leq 1$, is continuous for every $\omega \in \Theta$. Without loss of generality, from now on we can therefore assume that all trajectories of Brownian motion are continuous (by eventually replacing Ω by Θ).

Suppose

$$0 \leq t_1 < ... < t_n \leq 1$$

and let $I_1, ..., I_n$ be open subintervals of the real line. The set

$$S(t_1, ..., t_n; I_1, ..., I_n) = \{x \in C[0,1];\ x(t_k) \in I_k,\ k = 1, ..., n\}$$

is called an open n-cell in $C[0,1]$. A set in $C[0,T]$ is called an open cell if there exists an $n \in \mathbf{N}_+$ such that it is an open n-cell. The σ-algebra generated by all open cells in $C[0,1]$ is denoted by $\mathcal{C}$. The construction above shows that the map

$$W : \Omega \to C[0,1]$$

which maps ω to the trajectory

$$t \to W(t,\omega),\ 0 \leq t \leq 1$$

is $(\mathcal{F}, \mathcal{C})$-measurable. The image measure P_W is called Wiener measure in $C[0,1]$.

The Wiener measure is a Borel measure on the metric space $C\,[0,1]$. We leave it as an excersice to prove that

$$\mathcal{C} = \mathcal{B}(C\,[0,1]).$$

$\uparrow\uparrow\uparrow$