CHAPTER 6

COMPLEX INTEGRATION

Introduction

In this section, in order to illustrate the power of Lebesgue integration, we collect a few results, which often appear with uncomplete proofs at the undergraduate level.

6.1. Complex Integrand

So far we have only treated integration of functions with their values in **R** or $[0, \infty]$ and it is the purpose of this section to discuss integration of complex valued functions.

Suppose $(X, \mathcal{M}, \mu)$ is a positive measure. Let $f, g \in L^1(\mu)$. We define

$$\int_X (f + ig) d\mu = \int_X f d\mu + i \int_X g d\mu.$$

If α and β are real numbers,

$$\int_X (\alpha + i\beta)(f + ig) d\mu = \int_X ((\alpha f - \beta g) + i(\alpha g + \beta f)) d\mu$$

$$= \int_X (\alpha f - \beta g) d\mu + i \int_X (\alpha g + \beta f) d\mu$$

$$= \alpha \int_X f d\mu - \beta \int_X g d\mu + i\alpha \int_X g d\mu + i\beta \int_X f d\mu$$

$$= (\alpha + i\beta)(\int_X f d\mu + i \int_X g d\mu)$$

$$= (\alpha + i\beta) \int_X (f + ig) d\mu.$$

We write $f \in L^1(\mu; \mathbf{C})$ if Re f, Im $f \in L^1(\mu)$ and have, for every $f \in L^1(\mu; \mathbf{C})$ and complex α,

$$\int_X \alpha f d\mu = \alpha \int_X f d\mu.$$

Clearly, if $f, g \in L^1(\mu; \mathbf{C})$, then

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Now suppose μ is a complex measure on $\mathcal{M}$. If

$$f \in L^1(\mu; \mathbf{C}) =_{def} L^1(\mu_{\mathrm{Re}}; \mathbf{C}) \cap L^1(\mu_{\mathrm{Im}}; \mathbf{C})$$

we define

$$\int_X f d\mu = \int_X f d\mu_{\mathrm{Re}} + i \int_X f d\mu_{\mathrm{Im}}.$$

It follows for every $f, g \in L^1(\mu; \mathbf{C})$ and $\alpha \in \mathbf{C}$ that

$$\int_X \alpha f d\mu = \alpha \int_X f d\mu.$$

and

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

$\downarrow\downarrow\downarrow$

6.2. The Fourier Transform

Below, if $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n) \in \mathbf{R}^n$, we let

$$\langle x, y \rangle = \Sigma_{k=1}^n x_k y_k.$$

and

$$\mid x \mid = \sqrt{\langle x, y \rangle}.$$

If μ is a complex measure on $\mathcal{R}_n$ (or $\mathcal{R}_n^-$) the Fourier transform $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(y) = \int_{\mathbf{R}^n} e^{-i\langle x, y \rangle} d\mu(x), \ y \in \mathbf{R}^n.$$

Note that

$$\hat{\mu}(0) = \mu(\mathbf{R}^n).$$

The Fourier transform of a function $f \in L^1(m_n; \mathbf{C})$ is defined by

$$\hat{f}(y) = \hat{\mu}(y) \text{ where } d\mu = f dm_n.$$

Theorem 6.2.1. *The canonical Gaussian measure γ_n in $\mathbf{R}^n$ has the Fourier transform*

$$\hat{\gamma}_n(y) = e^{-\frac{|y|^2}{2}}.$$

PROOF. Since

$$\gamma_n = \gamma_1 \otimes ... \otimes \gamma_1 \text{ (n factors)}$$

it is enough to consider the special case $n = 1$. Set

$$g(y) = \hat{\gamma}_1(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}} \cos xy dx.$$

Note that $g(0) = 1$. Since

$$\mid \frac{\cos x(y+h) - \cos xy}{h} \mid \leq \mid x \mid$$

the Lebesgue Dominated Convergence Theorem yields

$$g'(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} -xe^{-\frac{x^2}{2}} \sin xy dx$$

(Exercise: Prove this by using Example 2.2.1). Now, by partial integration,

$$g'(y) = \frac{1}{\sqrt{2\pi}} \left[e^{-\frac{x^2}{2}} \sin xy \right]_{x=-\infty}^{x=\infty} - \frac{y}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}} \cos xy dx$$

that is

$$g'(y) + yg(y) = 0$$

and we get

$$g(y) = e^{-\frac{y^2}{2}}.$$

If $\xi = (\xi_1, ..., \xi_n)$ is an $\mathbf{R}^n$-valued random variable with $\xi_k \in L^1(P)$, $k = 1, ..., n$, the characteristic function c_ξ of ξ is defined by

$$c_\xi(y) = E\left[e^{i\langle \xi, y\rangle}\right] = \hat{P}_\xi(-y),\ y \in \mathbf{R}^n.$$

For example, if $\xi \in N(0, \sigma)$, then $\xi = \sigma G$, where $G \in N(0, 1)$, and we get

$$c_\xi(y) = E\left[e^{i\langle G, \sigma y\rangle}\right] = \hat{\gamma}_1(-\sigma y)$$

$$= e^{-\frac{\sigma^2 y^2}{2}}.$$

Choosing $y = 1$ results in

$$E\left[e^{i\xi}\right] = e^{-\frac{1}{2}E\left[\xi^2\right]} \text{ if } \xi \in N(0, \sigma).$$

Thus if $(\xi_k)_{k=1}^n$ is a centred real-valued Gaussian process

$$E\left[e^{i\Sigma_{k=1}^n y_k \xi_k}\right] = \exp(-\frac{1}{2}E\left[(\Sigma_{k=1}^n y_k \xi_k)^2\right]$$

$$= \exp(-\frac{1}{2}\Sigma_{k=1}^n y_k^2 E\left[\xi_k^2\right] - \Sigma_{1\leq j<k\leq n} y_j y_k E\left[\xi_j \xi_k\right]).$$

In particular, if

$$E\left[\xi_j \xi_k\right] = 0,\ j \neq k$$

we see that

$$E\left[e^{i\Sigma_{k=1}^n y_k \xi_k}\right] = \Pi_{k=1}^n e^{-\frac{y_k^2}{2}E\left[\xi_k^2\right]}$$

or

$$E\left[e^{i\Sigma_{k=1}^n y_k \xi_k}\right] = \Pi_{k=1}^n E\left[e^{iy_k \xi_k}\right].$$

Stated otherwise, the Fourier tranforms of the measures $P_{(\xi_1,...,\xi_n)}$ and $\times_{k=1}^n P_{\xi_k}$ agree. Below we will show that complex measures in $\mathbf{R}^n$ with the same Fourier transforms are equal and we get the following

Theorem 6.2.2. *Let $(\xi_k)_{k=1}^n$ be a centred real-valued Gaussian process with uncorrelated components, that is*

$$E\left[\xi_j \xi_k\right] = 0,\ j \neq k.$$

Then the random variables $\xi_1,...,\xi_n$ are independent.

6.3 Fourier Inversion

Theorem 6.3.1. *Suppose $f \in L^1(m_n)$. If $\hat{f} \in L^1(m_n)$ and f is bounded and continuous*

$$f(x) = \int_{\mathbf{R}^d} e^{i\langle y,x\rangle} \hat{f}(y) \frac{dy}{(2\pi)^n},\ x \in \mathbf{R}^n.$$

PROOF. Choose $\varepsilon > 0$. We have

$$\int_{\mathbf{R}^n} e^{i\langle y,x\rangle} e^{-\frac{\varepsilon^2}{2}|y|^2} \hat{f}(y) \frac{dy}{(2\pi)^n} = \int_{\mathbf{R}^n} f(u) \left\{ \int_{\mathbf{R}^n} e^{i\langle y,x-u\rangle} e^{-\frac{\varepsilon^2}{2}|y|^2} \frac{dy}{(2\pi)^n} \right\} du$$

where the right side equals

$$\int_{\mathbf{R}^n} f(u) \left\{ \int_{\mathbf{R}^n} e^{i\langle v,\frac{x-u}{\varepsilon}\rangle} e^{-\frac{1}{2}|v|^2} \frac{dv}{\sqrt{2\pi}^n} \right\} \frac{du}{\sqrt{2\pi}^n \varepsilon^n} = \int_{\mathbf{R}^n} f(u) e^{-\frac{1}{2\varepsilon^2}|u-x|^2} \frac{du}{\sqrt{2\pi}^n \varepsilon^n}$$

$$= \int_{\mathbf{R}^n} f(x+\varepsilon z) e^{-\frac{1}{2}|z|^2} \frac{dz}{\sqrt{2\pi}^n}.$$

Thus

$$\int_{\mathbf{R}^n} e^{i\langle y,x\rangle} e^{-\frac{\varepsilon^2}{2}|y|^2} \hat{f}(y) \frac{dy}{(2\pi)^n} = \int_{\mathbf{R}^n} f(x+\varepsilon z) e^{-\frac{1}{2}|z|^2} \frac{dz}{\sqrt{2\pi}^n}.$$

By letting $\varepsilon \to 0$ and using the Lebesgue Dominated Convergence Theorem, Theorem 6.3.1 follows at once.

Recall that $C_c^\infty(\mathbf{R}^n)$ denotes the class of all functions $f : \mathbf{R}^n \to \mathbf{R}$ with compact support which are infinitely many times differentiable. If $f \in C_c^\infty(\mathbf{R}^n)$ then $\hat{f} \in L^1(m_n)$. To see this, suppose $y_k \neq 0$ and use partial integration to obtain

$$\hat{f}(y) = \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle} f(x) dx = \frac{1}{iy_k} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle} f'_{x_k}(x) dx$$

and

$$\hat{f}(y) = \frac{1}{(iy_k)^l} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle} f^{(l)}_{x_k}(x) dx,\ l \in \mathbf{N}.$$

Thus

$$| y_k |^l | \hat{f}(y) | \leq \int_{\mathbf{R}^n} | f_{x_k}^{(l)}(x) | \, dx, \; l \in \mathbf{N}$$

and we conclude that

$$\sup_{y \in \mathbf{R}^n} (1+ | y |)^{n+1} | \hat{f}(y) | < \infty.$$

and, hence, $\hat{f} \in L^1(m_n)$.

Corollary 6.3.1. *If* $f \in C_c^\infty(\mathbf{R}^n)$, *then* $\hat{f} \in L^1(m_n)$ *and*

$$f(x) = \int_{\mathbf{R}^n} e^{i\langle y,x \rangle} \hat{f}(y) \frac{dy}{(2\pi)^n}, \; x \in \mathbf{R}^n.$$

Corollary 6.3.2 *If* μ *is a complex Borel measure in* $\mathbf{R}^n$ *and* $\hat{\mu} = 0$, *then* $\mu = 0$.

PROOF. Choose $f \in C_c^\infty(\mathbf{R}^n)$. We multiply the equation $\hat{\mu}(-y) = 0$ by $\frac{\hat{f}(y)}{(2\pi)^n}$ and integrate over $\mathbf{R}^n$ with respect to Lebesgue measure to obtain

$$\int_{\mathbf{R}^n} f(x) d\mu(x) = 0.$$

Since $f \in C_c^\infty(\mathbf{R}^n)$ is arbitrary it follows that $\mu = 0$. The theorem is proved.

6.4. Non-Differentiability of Brownian Paths

Let ND denote the set of all real-valued continuous function defined on the unit interval which are not differentiable at any point. It is well known that ND is non-empty. In fact, if ν is Wiener measure on $C\,[0,1]$, $x \in ND$ a.e. $[\nu]$. The purpose of this section is to prove this important property of Brownian motion.

Let $W = (W(t))_{0\le t\le 1}$ be a real-valued Brownian motion in the time interval $[0,1]$ such that every path $t \to W(t)$, $0 \le t \le 1$ is continuous. Recall that

$$E\left[W(t)\right] = 0$$

and

$$E\left[W(s)W(t)\right] = \min(s,t).$$

If

$$0 \le t_0 \le ... \le t_n \le 1$$

and $1 \le j < k \le n$

$$E\left[(W(t_k) - W(t_{k-1}))(W(t_j) - W(t_{j-1})\right]$$

$$= E\left[(W(t_k)W(t_j)\right] - E\left[W(t_k)W(t_{j-1})\right] - E\left[W(t_{k-1})W(t_j)\right] + E\left[W(t_{k-1})W(t_{j-1})\right]$$

$$= t_j - t_{j-1} - t_j + t_{j-1} = 0.$$

From the previous section we now infer that the random variables

$$W(t_1) - W(t_0), ..., W(t_n) - W(t_{n-1})$$

are independent.

Theorem 7. *The function* $t \to W(t)$, $0 \le t \le 1$ *is not differentiable at any point* $t \in [0,1]$ a.s. $[P]$.

PROOF. Without loss of generality we assume the underlying probability space is complete. Let $c, \varepsilon > 0$ and denote by $B(c,\varepsilon)$ the set of all $\omega \in \Omega$ such that

$$\mid W(t) - W(s) \mid < c \mid t - s \mid \text{ if } t \in [s-\varepsilon, s+\varepsilon] \cap [0,1]$$

for some $s \in [0,1]$. It is enough to prove that the set

$$\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} B(j, \frac{1}{k}).$$

is of probability zero. From now on let $c, \varepsilon > 0$ be fixed. It is enough to prove $P\left[B(c,\varepsilon)\right] = 0$.

Set

$$X_{n,k} = \max_{k \le j < k+3} \mid W(\frac{j+1}{n}) - W(\frac{j}{n}) \mid$$

for each integer $n > 3$ and $k \in \{0, ..., n-3\}$.

Let $n > 3$ be so large that

$$\frac{3}{n} \le \varepsilon.$$

We claim that

$$B(c, \varepsilon) \subseteq \left[\min_{0 \le k \le n-3} X_{n,k} \le \frac{6c}{n} \right].$$

If $\omega \in B(c, \varepsilon)$ there exists an $s \in [0, 1]$ such that

$$\mid W(t) - W(s) \mid \le c \mid t - s \mid \text{ if } t \in [s - \varepsilon, s + \varepsilon] \cap [0, 1].$$

Now choose $k \in \{0, ..., n-3\}$ such that

$$s \in \left[\frac{k}{n}, \frac{k}{n} + \frac{3}{n} \right].$$

If $k \le j < k+3$,

$$\mid W(\frac{j+1}{n}) - W(\frac{j}{n}) \mid \le \mid W(\frac{j+1}{n}) - W(s) \mid + \mid W(s) - W(\frac{j}{n}) \mid$$

$$\le \frac{6c}{n}$$

and, hence, $X_{n,k} \le \frac{6c}{n}$. Now

$$B(c, \varepsilon) \subseteq \left[\min_{0 \le k \le n-3} X_{n,k} \le \frac{6c}{n} \right]$$

and it is enough to prove that

$$\lim_{n \to \infty} P \left[\min_{0 \le k \le n-3} X_{n,k} \le \frac{6c}{n} \right] = 0.$$

But

$$P \left[\min_{0 \le k \le n-3} X_{n,k} \le \frac{6c}{n} \right] \le \sum_{k=0}^{n-3} P \left[X_{n,k} \le \frac{6c}{n} \right]$$

$$= (n-2)P\left[X_{n,0} \leq \frac{6c}{n}\right] \leq nP\left[X_{n,0} \leq \frac{6c}{n}\right]$$

$$= n(P\left[\mid W(\frac{1}{n}) \mid \leq \frac{6c}{n}\right])^3 = n(P(\mid W(1) \mid \leq \frac{6c}{\sqrt{n}})^3$$

$$\leq n(\frac{12c}{\sqrt{2\pi n}})^3.$$

where the right side converges to zero as $n \to \infty$. The theorem is proved.

Recall that a function of bounded variation possesses a derivative a.e. with respect to Lebesgue measure. Therefore, with probability one, a Brownian path is not of bounded variation. In view of this an integral of the type

$$\int_0^1 f(t)dW(t)$$

cannot be interpreted as an ordinary Stieltjes integral. Nevertheless, such an integral can be defined by completely different means and is basic in, for example, financial mathematics.

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