Matematiska vetenskaper
CTH & GU
P. Sjögren 2006

Övningar i FUNKTIONALANALYS

1. Consider the space $X = C^1[0,1]$ of continuously differentiable functions on $[0,1]$ with one-sided derivatives at the endpoints. Show that X is not a Banach spaces if it is given the norm $\|f\| = \sup_{[0,1]} |f|$. Cf. Folland exerc. 5.1.9 p. 155.

2. Let X be a normed vector space. Prove the inequality $|\,\|x\| - \|y\|\,| \le \|x - y\|$. Conclude that the norm is continuous on X, in the sense that $x_j \to x$ implies $\|x_j\| \to \|x\|$. Cf. Folland exerc. 5.1.1 p. 154.

3. Visa att ett ändligt-dimensionellt delrum av ett normerat linjärt rum är slutet. Ledning: kalla delrummet V, och anta att en följd ur V konvergerar mot någon vektor i X. Då är följden Cauchy och därför konvergent i V.

4. If equality holds in the triangle inequality of a normed space, do the vectors have to be parallel? More precisely, does $\|x + y\| = \|x\| + \|y\|$ imply that one of the vectors is a scalar multiple of the other? Answer the question for ℓ^1, ℓ^2 and ℓ^∞.

5. Let $w = (w_k)$ be a sequence of positive numbers. Define a measure m_w on $\mathbb{N}$ by placing the mass w_k at k for $k = 1, 2, \dots$. Then $L^p(m_w)$ is written ℓ^p_w and called weighted ℓ^p. Prove that ℓ^p_w is isometrically isomorphic with ℓ^p for $1 \le p \le \infty$.

6. Consider real, two-dimensional ℓ^p, which is $\mathbb{R}^2$ with the norm
$$\|(x_1, x_2)\| = (|x_1|^p + |x_2|^p)^{1/p}, \qquad 1 \le p < \infty,$$
with the usual interpretation for $p = \infty$. Sketch the unit ball for several values of p, for instance $p = 1,\ 3/2,\ 2,\ 3,\ \infty$. For which p is the unit ball strictly convex, in the sense that $\|x\|, \|y\| \le 1$ and $x \ne y$ imply $\|(x + y)/2\| \le 1$?

7. (Packing of balls) Place inside the unit ball of ℓ^p, $1 \le p \le \infty$, infinitely many pairwise disjoint balls having the same positive radius. Conclude that the (closed) unit ball cannot be compact (for the definition of compact, see the following problem).

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8. Låt M vara ett fullständigt metriskt rum och K en delmängd av M. Visa att följande är ekvivalent.

   (i) K är kompakt, dvs. varje öppen övertäckning av K har en ändlig delövertäckning.

   (ii) K är följdkompakt, dvs. varje följd i K har en delföljd som konvergerar mot en punkt i K.

   (iii) K är sluten och dessutom totalt begränsad, dvs man kan för varje $r > 0$ täcka K med ändligt många klot av radie r.

9. (Generalisering av övning 7) Låt X vara ett oändligtdimensionellt normerat rum.
   a) Om $r > 0$ är litet, visa att man i enhetsklotet i X kan placera oändligt många parvis disjunkta klot av radie r. Använd t ex Follands övning 5.1.12(b) sid 156, kombinerad med övning 3 ovan.
   b) Dra slutsatsen att ett klot i X aldrig är kompakt.

10. (Extension by continuity) Let M be a dense subspace of the normed space X. Here dense means that $\overline{M} = X$. Let Y be a Banach space. Show that any bounded linear operator $T : M \to Y$ has a unique extension to a bounded linear operator $X \to Y$.

11. a) Prove that $\|.\|_p$ is not a norm for $0 < p < 1$, using, say, Lebesgue measure in $[0, 1]$.
    b) Prove that it is not even equivalent to a norm.
    Hint: Use characteristic functions of disjoint sets.

12. Let $1 \leq p < s < r \leq \infty$. Prove the inclusions

$$L^p \cap L^r \subset L^s \subset L^p + L^r.$$

The measure is arbitrary. Compare with Folland, exerc. 6.1.3,4 p. 186.

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13. Consider convergence in L^p norm with $1 \le p < \infty$ and a.e. convergence, for some measure of your choice. Prove that neither implies the other.

14. With $1 \le p < \infty$, assume that the function sequence f_j converges to f in L^p for some measure. Prove that there is a subsequence which converges a.e. to f.

15. Assume that the function g is measurable and a.e. finite, with respect to Lebesgue measure, but that $g \notin L^q$. Here $1 < q < \infty$, and we let p denote the dual exponent. Find a function $f \in L^p$ for which $fg \notin L^1$.

16. Prove that ℓ^∞ and $L^\infty[0,1]$ are not separable. Hint: Find an uncountable set of points, all at mutual distances at least r, for some $r > 0$. In the first case, cf. exercise 7.

17. Let E be the subset of ℓ^p, $1 \le p \le \infty$, consisting of all sequences with only finitely many nonzero entries. Determine the closure $\overline{E}$.

18. Translation is continuous in $L^p(\mathbb{R})$ for $1 \le p < \infty$: If $f_t(x) = f(x-t)$ with $f \in L^p$, then $f_t \to f_{t'}$ in L^p as $t \to t'$. Prove also that this is false for $p = \infty$.

19. Are the normed spaces $L^p(\mathbb{R})$ and $L^p[0,1]$ isometrically isomorphic? Here $1 \le p \le \infty$.

20. For which sequences (d_k) of positive numbers is

$$\{(x_k) \in \ell^p : |x_k| \le d_k,\ k = 1, 2, ...\}$$

a compact subset of ℓ^p? Here $1 \le p \le \infty$. As for compactness, see problem 8.

21. The map $(s, 0) \mapsto s$ is a linear functional on the subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^2$. Determine all its linear extensions to $\mathbb{R}^2$ with the same norm, if $\mathbb{R}^2$ is given (a) the ℓ^1 norm or (b) the ℓ^∞ norm.

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22. (Separable case of Hahn-Banach) Let X be a separable normed space. Assume that M is a subspace of X and that f is a linear functional defined on M and bounded with respect to the norm of X. Prove that f has an extension to a bounded linear functional on X, without using anything equivalent to the axiom of choice.

23. A linear subspace M of a normed space X is called a hyperplane if for some (and hence all) $x \in X \setminus M$ the subspace M and the vector x together span X. One says that the codimension of M is 1.
(a) Prove that a hyperplane is either closed or dense in X.
(b) Prove that any hyperplane is given by $M = f^{-1}(0)$ (the kernel or zero space of f) for some linear functional on X. Also show that M is closed precisely when f is continuous.
(c) If two linear functionals have the same kernel, prove that they are proportional.

24. Let X be a Banach space of infinite dimension. Prove that X has no countable "basis", if we let this mean that any vector can be written as a finite linear combination of vectors from the "basis".

25. Baire category is not related to Lebesgue measure. Find a meager subset of $\mathbb{R}$ whose complement has measure 0. Hint: Construct open dense subsets of arbitrarily small measure. A countable intersection of such sets can have measure 0.

26. Consider the boundary of an open subset U of the real line. Is ∂U always nowhere dense? Is ∂U always of Lebesgue measure 0?

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27. Prove that the principle of uniform boundedness does not hold in all normed linear spaces.

28. Let $\mathbb{Q} = (q_j)_1^\infty$ be an enumeration of the rational numbers, and let $\epsilon_{n,j}$, $n, j = 1, 2, ...$ be arbitrary positive numbers. Then the set
$$\bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} (q_j - \epsilon_{n,j}, q_j + \epsilon_{n,j})$$
obviously contains $\mathbb{Q}$. Prove that this inclusion is always strict.

29. Show that a linear isometry T from a Hilbert space into another preserves the inner product, in the sense that $\langle Tx, Ty\rangle = \langle x, y\rangle$.

30. Consider the function sequence $(e^{inx})_n$ in $L^p(-\pi, \pi)$. For which p, $1 \le p \le \infty$, is it weakly or weakly* convergent?

31. Let (e_n) denote the "standard basis" in ℓ^p. For which p with $1 \le p \le \infty$ does e_n converge weakly or weakly* as $n \to \infty$, and what is the limit?

32. Assume that $f_n \to f$ weakly in a normed space, or weakly* in its dual. Prove that
$$\|f\| \le \underline{\lim}\, \|f_n\|.$$
Also give examples where this inequality is strict.

33. In L^∞ translation is weakly* continuous: With the notation from problem 18, show that $f_t \to f_{t'}$ weakly* in $L^\infty = (L^1)^*$, as $t \to t'$.

34. We know that $\ell^1 = c_0^*$ and $(\ell^1)^* = \ell^\infty$. Give an example of a sequence of vectors in ℓ^1 that converges weakly* but not weakly, which means that $\langle f, x_n\rangle \to \langle f, x\rangle$ for all $f \in c_0$ but not for all $f \in \ell^\infty$.

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35. Let μ be a Radon measure in a locally compact Hausdorff space X. (This means $\mu \geq 0$.) Take $1 \leq p < \infty$. Prove that the space of simple functions is dense in $L^p(\mu)$. By a simple function we mean a finite linear combination of characteristic functions of Borel sets with finite measure. Also prove that $C_c(X)$ is dense in $L^p(\mu)$.

36. Let X be a locally compact Hausdorff space. Assume that the signed Radon measures μ_n in X converge vaguely, i.e., weakly* in $M(X) = C_0^*$, to μ. If in addition it is assumed that X is compact and that the μ_n are positive, prove that $\|\mu_n\| \to \|\mu\|$. Also prove that this need not hold if either of the two additional assumptions is suppressed.

37. In the locally compact Hausdorff space X, assume that the (positive) finite Radon measures $\mu_n \in M(X) = C_0^*$ converge vaguely to μ. Give an example to show that $\mu_n(A)$ need not tend to $\mu(A)$ for a Borel set $A \subset X$. Prove that, however, $\mu(U) \leq \underline{\lim}\, \mu_n(U)$ for each open set U.

38. In the unit disc, prove that the positive harmonic functions are precisely the Poisson integrals of the (positive) Borel measures on the unit circle.

39. Consider the space weak L^p for some measure and some $p > 0$. Show that the quantity $[f]_p$ has all the properties of a norm, except that the triangle inequality is violated. Also prove that instead of the triangle inequality one has $[f+g]_p \leq C([f]_p+[g]_p)$ for some constant C; one speaks of a *quasinorm*. Cf. here Folland's exercise 6.4.35 p 199.

40. With $1 \leq p, r, s \leq \infty$, we want to see that the equation $1/s = 1/p + 1/r - 1$ is necessary for Young's inequality. Consider the line with Lebesgue measure, and assume that the equation does not hold.

(a) Show that there is no norm inequality

$$\|f * g\|_s \leq C\|f\|_p \|g\|_r$$

for any C.

(b) Find a counterexample to the inclusion $L^p * L^r \subset L^s$.