MATEMATIK
Göteborg University

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Tentamensskrivning in Diskret Matematik

The questions can be answered in English or in Swedish.

1. On the set $\mathbf{N} \times \mathbf{N}$ define the relation $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \leq d$. Show that this defines a reflexive antisymmetric and transitive relation. How many elements (a,b) satisfy $(a,b) \leq (2,3)$? In general how many elements (a,b) satisfy $(a,b) \leq (p,q)$? Show that the relation $(a,b) < (c,d)$ - defined as usual by $(a,b) \leq (c,d)$ and $(a,b) \neq (c,d)$ - is well-founded.

2. Let E be the set of infinite subsets X of $\mathbf{N}$ such that the complement $\mathbf{N} - X$ of X is also infinite. Give an example of an element of E. We define the relation $X < Y$ on E by: $X \subseteq Y$ and $X \neq Y$. Show that $<$ is not well-founded by giving an example of an infinite chain $X_0 > X_1 > X_2 \ldots$

3. Give the definition of a *countable* set and the proof that the set $\{0,1\}^{\mathbf{N}}$ is *not* countable.

4. Find how many (i) functions (ii) one-to-one functions (iii) onto functions (iv) bijections there are from X to Y in these cases; in each case, justify your answer:

   - $X = \{1,2\},\ Y = \{1,2\}$,
   - $X = \{1,2,3,4,5\},\ Y = \{1,2\}$,
   - $X = \{1,2\},\ Y = \{1,2,3,4,5,6,7,8,9\}$

5. Classify each of the following propositions as a tautology, a contradiction or neither. Justify your claims by using truth tables or otherwise.

   (a) $\neg P \to P$, (b) $((P \to Q) \to P) \to P$, (c) $P \vee (P \to Q)$,
   (d) $(P \vee Q) \to (P \wedge Q)$,

   and finally

   (e) $((P \vee Q) \wedge (Q \vee R) \wedge (R \vee P)) \to ((P \wedge Q) \vee (Q \wedge R) \vee (R \wedge P))$.

6. In a boolean algebra, prove the following equivalence

   (1) $a = b \Longleftrightarrow (a \wedge b') \vee (a' \wedge b) = 0$

   (2) $a \wedge b \leq c \vee d \Longleftrightarrow a \wedge c' \leq b' \vee d$.

7. Apply the unification algorithm on the following pair of terms, with $C_3 = \{f\},\ C_1 = \{g\}$ (C_n is the set of function symbols of arity n):

   (a) $f(g(y), g(x), g(z))$ and $f(g(x), g(z), g(g(y)))$,
   (b) $f(f(z,z,x), y, x)$ and $f(y,y,z)$.

8. Let (a_n) and (b_n) be two sequences satisfying $a_0 = b_0 = 1$ and

$$a_{n+1} - b_n = 0$$

$$b_{n+1} - a_n - b_n = 0.$$

Find a recurrence equation for (a_n) of the form $a_{n+2} = Aa_{n+1} + Ba_n$. Give a closed form formula for a_n and b_n.

9. Which one of these formulae are valid? In each case, motivate your answer:

(a) $[\neg\forall x\ P(x)] \Rightarrow [\forall x \neg P(x)]$,

(b) $[\exists x \neg P(x)] \Rightarrow [\neg\exists x\ P(x)]$,

(c) $[\neg\exists x\ P(x)] \Rightarrow [\forall x \neg P(x)]$.