Solutions to the first homework

1. From Hein's book

   2.3.6 It is straightforward that $x/1-x = y$ iff $x = y/1+y$. Hence f has an inverse $y \mapsto y/1+y$ and hence is a bijection. Similarly g has an inverse $y \mapsto 1/1+y$ and so is a bijection.

   2.3.10 a. We can tell only that g is surjective. b. We can tell only that f is injective.

   2.3.11 a. See Hein, page 783. b. Let g be injective. Suppose $a_1, a_2 \in A$ and $a_1 \neq a_2$. In this case $g(a_1) \neq g(a_2)$, which implies $\langle g(a_1), h(a_1)\rangle \neq \langle g(a_2), h(a_2)\rangle$, i.e. $f(a_1) \neq f(a_2)$. Hence f is injective. Example: $A = \{1,2,3\}, B = \{a,b\}$ and $C = \{c,d\}$. If $g = \{1 \mapsto a, 2 \mapsto a, 3 \mapsto b\}$ and $h = \{1 \mapsto c, 2 \mapsto d, 3 \mapsto d\}$, then neither g nor h is injective. However, f is indeed injective.

2. If $f : X \to Y$ is a function from X to Y and A, B are subsets of X show that $f(A \cap B) \subseteq f(A) \cap f(B)$. Give an example where we have $f(A \cap B) \neq f(A) \cap f(B)$. Show however that, if f is *injective* then we have $f(A \cap B) = f(A) \cap f(B)$.

   Solution: $y \in f(A \cap B) \Rightarrow \exists x \in A \cap B . f(x) = y \Rightarrow \exists x \in A . f(x) = y \& \exists x \in B . f(x) = y \Rightarrow y \in f(A) \cap f(B)$. Example: $A = \{1\}, B = \{2\}$ and $f(1) = f(2) = a$. Further, $y \in f(A) \cap f(B) \Rightarrow \exists x_1 \in A, x_2 \in B . f(x_1) = f(x_2) = y$. If f is injective, then $x_1 = x_2$ and $x_1 \in A \cap B$. Thus $y \in f(A \cap B)$.

3. What are the elements of

   $$X = \{(A,B) \mid A \subseteq B \subseteq E\}$$

   for $E = \{1\}$? For $E = \{1,2\}$? In general, if $|E| = n$ show that

   $$|X| = \Sigma_{0 \leq k \leq n} \binom{n}{k} 2^k$$

   by using a suitable partition $X_k,\ 0 \leq k \leq n$ of X. Is there a simpler way to write $|X|$?

   Solution: For $E = \{1\}$ we have

   $$X = \{(\emptyset,\emptyset), (\emptyset, E), (E,E)\}$$

   and for $E = \{1,2\}$ we have

   $$X = \{(\emptyset,\emptyset), (\emptyset,\{1\}), (\emptyset,\{2\}), (\emptyset,E), (\{1\},\{1\}), (\{1\},E), (\{2\},\{2\}), (\{2\},E), (E,E)\}.$$

   If we take $X_k = \{(A,B) \in X \mid |B| = k\}$. It is clear that X_k's are a partition of X for $0 \leq k \leq n$. Furthermore, there are $\binom{n}{k}$ choices for B, and for each choice of B 2^k choices of A in building an element of X_k. Hence $|X_k| = \binom{n}{k} 2^k$ and so

   $$|X| = \Sigma |X_k| = \Sigma_{0 \leq k \leq n} \binom{n}{k} 2^k.$$

   By the binomial theorem, this can be written simply as $|X| = 3^n$.

4. Let $A = \{1, \ldots, 8\}$. Remark that there exists a partition of A in four subsets A_1, A_2, A_3, A_4 such that $|A_i| = 2$ and the sum of elements of each A_i is 9. Deduce that if we take 5 elements a_1, a_2, a_3, a_4, a_5 of A then we can find $i < j$ such that $a_i + a_j = 9$.

   Solution: We take $A_i = \{i, 9 - i\}$. By the pigeon-hole principle, two of the five elements a_1, a_2, a_3, a_4, a_5 are in the same A_k. If we write these elements a_i and a_j we have $a_i, a_j \in A_k$ and by construction $a_i + a_j = 9$.

5. (More difficult, and not required for points) We recall that $(Z/2Z, 0, +)$ is the commutative group of integers *modulo* 2, where $x = y\ (mod.2)$ iff 2 divides $x - y$. Let U be a set. If $A \oplus B$ is the symmetric difference on $pow(U)$ show that $\chi_{A \oplus B}(x) = \chi_A(x) + \chi_B(x)\ (mod.2)$ and deduce a new proof that the operation $\oplus$ is commutative and associative. Notice then that $x \in A_1 \oplus \ldots \oplus A_n$ iff x belongs to an *odd* number of A_i.

   Let now (V, E) be a non oriented graph, that is V is a set and E is a set of pairs $\{x, y\}$ of elements in V. If $e_1, \ldots, e_n$ form a path

$$e_1 = \{x_0, x_1\},\ e_2 = \{x_1, x_2\}, \ldots,\ e_n = \{x_{n-1}, x_n\}$$

   prove that $e_1 \oplus \ldots e_n = \{x_0, x_n\}$ if $x_0 \neq x_n$ and $e_1 \oplus \ldots e_n = \emptyset$ if $x_0 = x_n$. Deduce from all this a proof of one direction of Euler's theorem: if there is an Eulerian circuit then each degree has to be even, and if there is an Eulerian path between two vertices then these two vertices are the only one with an odd degree.

   Solution: One can see $\chi_{A \oplus B}(x) = \chi_A(x) + \chi_B(x)\ (mod\ 2)$ by studying the four cases determined by whether $x \in A$ and $x \in B$. Operation $\oplus$ is commutative and associative because so is $+\ (mod\ 2)$.

   If there is a Eulerian circuit then $e_1 \oplus \ldots e_n = \emptyset$, which implies that each of the x_i's belongs to an even number of e_i's. Thus, each vertice' degree has to be even. Similarly, if there is a Eulerian path between two vertices then these two vertices are the only ones with an odd degree.