A Matrix inequality

This little calculation is only supposed to compensate for a rather unclear explanation in class ...

Let $u = (u_1, ..., u_n)^T$ and $w = (w_1, ..., w_N)^T$ be vectors in $\mathbb{R}^n$, and let $A = (a_{ij})$ be an $n \times n$-matrix. Morover, let the $|u|$ denote the usual Euclidian norm

$$|u| \quad = \quad \sqrt{\sum_{k=1}^{n} u_k^2}$$

Recall the triangle inequality:

$$|u + w| \quad \leq \quad |u| + |w|\,,$$

and that there is equality in this expression only if u and w are parallel.

The inequality that I wanted to prove was the following:

$$|Aw| \quad \leq \quad n \max_{i,j} |a_{i,j}||w|\,. \tag{1}$$

(in fact, in the lecture, the matrix coefficients a_{ij} were allowed to depend on x, and in that case the maximum value must be taken also over all x in the appropriate interval).

Written in full detail, equation (1) becomes

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} a_{11}w_1 + a_{12}w_2 + \cdots a_{1n}w_n \\ a_{21}w_1 + a_{22}w_2 + \cdots a_{2n}w_n \\ \vdots \\ a_{n1}w_1 + a_{n2}w_2 + \cdots a_{nn}w_n \end{bmatrix}$$

Now note that the right hand side can be written

$$\begin{bmatrix} a_{11}w_1 \\ a_{22}w_2 \\ a_{33}w_3 \\ \vdots \\ a_{nn}w_n \end{bmatrix} + \begin{bmatrix} a_{12}w_2 \\ a_{23}w_3 \\ a_{34}w_4 \\ \vdots \\ a_{n1}w_1 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}w_n \\ a_{21}w_1 \\ a_{32}w_2 \\ \vdots \\ a_{n(n-1)}w_{n-1} \end{bmatrix} \tag{2}$$

(you should verify that all terms appear exactly once). Now,

$$\left\| \begin{bmatrix} a_{12}w_2 \\ a_{23}w_3 \\ a_{34}w_4 \\ \vdots \\ a_{n1}w_1 \end{bmatrix} \right\| \quad \leq \quad \max_{i,j} |a_{ij}| \left\| \begin{bmatrix} w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_1 \end{bmatrix} \right\|$$

and that if the the norm of a vector does not change if one permutes the vector components:

$$\left\| \begin{bmatrix} w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_1 \end{bmatrix} \right\| \quad = \quad \left\| \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} \right\|$$

The same reasoning applies to all terms in (2), and hence that all n terms can be estimated by the same expression,

$$\max_{i,j} |a_{ij}||w|\,,$$

and this is enough to prove that the inequaltiy (1) holds.