## Former Examination 1

Admitted: Arbitrary calculator, "accepted" mathematical handbook, e.g. Beta If you want to use another handbook then, please, CONTACT ME AT LEAST ONE WEEK BEFORE THE EXAMINATION
Responsible for the course: Johannes Brasche

1. Formulate and prove the statement about uniqueness in "Picards theorem".

2 . Let $A:(a, b) \rightarrow \mathbb{R}^{n \times n}$ be continuous.
(a) Prove that the initial value problem

$$
\underline{y}^{\prime}(t)=A \cdot \underline{y}(t), \quad \underline{y}\left(t_{0}\right)=\underline{y}_{0}
$$

has exactly one solution for every $t_{0} \in(a, b)$ and every $\underline{y}_{0} \in \mathbb{R}^{n}$.
(b) Prove that $\underline{y}^{\prime}(t)=A \cdot \underline{y}(t)$ has $n$ solutions $\underline{y}_{1}, \underline{y}_{2}, \ldots \underline{y}_{n}$ such that for every $t \in(a, b)$ the vectors

$$
\underline{y}_{1}(t), \underline{y}_{2}(t), \ldots, \underline{y}_{n}(t)
$$

are linearly independent.
3. Determine a solution different from $u(t) \equiv 0$ of

$$
y^{\prime}(t)=t \sqrt{y(t)}, \quad y(0)=0 .
$$

4. Let $A=\left(\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1\end{array}\right)$ and $\underline{y}_{0}=\left(\begin{array}{l}2 \\ 1 \\ 0\end{array}\right)$. Determine the solution of

$$
\underline{y}^{\prime}(t)=A \cdot \underline{y}(t), \underline{y}(0)=\underline{y}_{0} .
$$

5. Let

$$
T g(t):=\int_{0}^{t} 2 s(1+g(s)) d s, \quad t \in \mathbb{R}
$$

for all continuous functions $g: \mathbb{R} \rightarrow \mathbb{R}$.
(a) For $g(t) \equiv 0$ compute $T g, T^{2} g$ and $T^{3} g$.
(b) For $g(t) \equiv 0$ give a formula for $T^{n} g$ and prove it.
6. This exercise is NOT relevant in the examination in 2003

Let

$$
\begin{equation*}
y^{\prime}(t)=\lambda y(t), y(0)=1 \tag{1}
\end{equation*}
$$

Let $h>0$.
(a) Which values for $\tilde{y}(h)$ and $\tilde{y}(2 h)$ does one get if if one computes an approximative solution of (1) with the aid of the forward trapetz method.
(b) Describe even the relation between

$$
\tilde{y}((j+1) h) \quad \text { och } \quad \tilde{y}(j h)
$$

for arbitrary $j \in \mathbb{N}$.
7. Prove that the problem

$$
y^{\prime \prime}(t)+\alpha y^{\prime}(t)+\beta y(t)=F(t), \quad y\left(t_{0}\right)=y_{0}, \quad y^{\prime}\left(t_{0}\right)=y_{0}^{(1)},
$$

has exactly one solution for every $\alpha, \beta, t_{0}, y_{0}, y_{0}^{(1)} \in \mathbb{R}$ and every continuous function $F: \mathbb{R} \rightarrow \mathbb{R}$.
8. (a) Let $y^{\prime \prime}(t)=-V^{\prime}(y(t)), t \in \mathbb{R}$.

Put

$$
E(t):=\frac{1}{2} y^{\prime}(t)^{2}+V(y(t)), t \in \mathbb{R} .
$$

Show that the function $E$ is constant.
(b) Let

$$
\begin{equation*}
y^{\prime \prime}(t)+2 y(t) e^{-y(t)^{2}}=0, t \in \mathbb{R} . \tag{2}
\end{equation*}
$$

Give a function $V: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
V(0)=-1 \quad \text { och } \quad y^{\prime \prime}(t)=-V^{\prime}(y(t)), t \in \mathbb{R}
$$

(c) Let $y$ be a solution of (2) satisfying $y(0)=y_{0}$ and $y^{\prime}(0)=0$. Prove the following:
If $y(t)=0$ then $y^{\prime}(t)^{2}=2\left(1-e^{-y_{0}^{2}}\right)$.

In the course corresponding to this old examination we did not treat boundary value problems (but, instead of this, numerics). In 2003 there will be also exercises on boundary value problems, cf. the other old examination.

