MAN460 och TMA013
Examination in “Ordinary differential equations”

Mathematics, CTH & GU, June 04, 2004, 8.45 – 13.45
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Auxiliary tools: Admitted calculator, admitted mathematical handbook (e.g.Beta)
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1. Let $-\infty < a < b < \infty$ and $S = \{(t,x) : a \leq t \leq b, x \in \mathbb{R}\}$. Let $f : S \longrightarrow \mathbb{R}$ be continuous and suppose that there exists a constant L such that

$$|f(t,x) - f(t,z)| \leq L|x - z|$$

for all $x, z \in \mathbb{R}, a \leq t \leq b$ and $L \cdot (b - a) < 1/2$. Let $\eta \in \mathbb{R}$. Show that the initial value problem

$$\begin{aligned} y'(t) &= f(t, y(t)), \\ y(a) &= \eta, \end{aligned}$$

has exactly one solution.

Solution: See the course books.

2. Let $\phi : [0, a] \longrightarrow \mathbb{R}$ be continuous, $\alpha \in \mathbb{R}$, $\beta > 0$ and

$$\phi(t) \leq \alpha + \beta \int_0^t \phi(s)ds$$

for all $0 \leq t \leq a$. Then

$$\phi(t) \leq \alpha e^{\beta t}$$

for all $0 \leq t \leq a$ (Grönvall's lemma). Prove this lemma.

Solution: See the course books.

3. Solve the following initial value problem:

$$y'(t) = \frac{e^{-y^2(t)}}{y(t)(2t + t^2)}, \quad y(2+) = 0.$$

Solution:

$$\int ye^{y^2}dy = \int \frac{1}{2t+t^2}dt = \frac{1}{2}(\ln(\frac{t}{t+2}) + c).$$

$$\int ye^{y^2}dy = \frac{1}{2}e^{y^2}.$$

Thus

$$y^2 = \ln(\ln(\frac{t}{t+2}) + c),$$

$$y(t) = \sqrt{\ln(\ln(\frac{t}{t+2}) + c)}.$$

$$y(2+) = \{\ln[\ln(\frac{1}{2}) + c]\}^{1/2} = 0.$$

Thus

$$\ln(\frac{1}{2}) + c = 1, \quad c = 1 - \ln(\frac{1}{2}) = 1 + \ln(2).$$

4. Is the solution to the initial value problem

$$y'(t) = \sqrt{|y(t)|}, \quad y(0) = 0,$$

unique? If not, then give two different solutions.

Solution: Obviously $y \equiv 0$ is a solution. We try to find another solution via the method of separation of variables.

$$\int \frac{dy}{\sqrt{|y|}} = \int dt = t + 2c.$$

For $y > 0$ we have

$$\int y^{-1/2}dy = 2y^{1/2} = t + 2c, \quad y(t) = (t/2 + c)^2.$$

$y(0) = 0$ yields $c = 0$.

The function

$$y(t) = \begin{cases} t^2/4, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

is another solution (note that the derivative of $t^2/4$ at $t = 0$ equals zero such that the above function really is differentiable at $t = 0$).

5. a) Determine two linearly independent real solutions to

$$\begin{aligned} x_1'(t) &= 3x_1(t) + 6x_2(t), \\ x_2'(t) &= -2x_1(t) - 3x_2(t). \end{aligned}$$

Solution: a) *We determine the eigenvalues of the coefficient matrix A*

$$\det(A - \lambda I) = (3-\lambda)(-3-\lambda) - 6(-2) = \lambda^2 + 3 = 0 \Longleftrightarrow \lambda = \sqrt{3}\, i.$$

Thus A has the eigenvalues $\pm\sqrt{3}\, i$.

b) *We compute an eigenvector* $\mathbf{u}$ *corresponding to the eigenvalue* $\sqrt{3}\, i$

$$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \sqrt{3}\, i \cdot \begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow 3x + 6y = \sqrt{3}\, i\, x \Longrightarrow x = \frac{6}{\sqrt{3}\, i - 3} y = (-\frac{3}{2} - \frac{\sqrt{3}}{2}\, i) y.$$

Thus

$$\mathbf{u} = \begin{bmatrix} -3/2 - \sqrt{3}\, i/2 \\ 1 \end{bmatrix}$$

is an eigenvector of A corresponding to the eigenvalue $\sqrt{3}\, i$.

c) *We give a complex solution*

$$\begin{aligned} u(t) &= e^{\sqrt{3}\, i\, t} \cdot \begin{bmatrix} -3/2 - \sqrt{3}\, i/2 \\ 1 \end{bmatrix} \\ &= (\cos(\sqrt{3}t) + i \sin(\sqrt{3}t)) \cdot \begin{bmatrix} -3/2 - \sqrt{3}\, i/2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -3/2 \cos(\sqrt{3}\, t) + \sqrt{3}/2 \sin(\sqrt{3}\, t) \\ \cos(\sqrt{3}\, t) \end{bmatrix} + i \begin{bmatrix} -3/2 \sin(\sqrt{3}\, t) - \sqrt{3}/2 \cos(\sqrt{3}\, t) \\ \sin(\sqrt{3}\, t) \end{bmatrix} \end{aligned}$$

is a complex solution. The real part and the imaginary part of u are then two linearly independent real solutions.

b) Is $x(t) \equiv 0$ a stable solution of this system of differential equations?

Solution: Yes, it is. Since the coefficient matrix has two linearly independent eigenvectors and the maximum over the real parts of the eigenvalues is less than or equal to 0, all solutions are stable.

6. a) Determine all real numbers λ_n such that

$$-u'' = \lambda_n u$$

has a solution $u_n \not\equiv 0$ on $[0, \pi]$ satisfying the Dirichlet boundary conditions $u_n(0) = 0 = u_n(\pi)$.

Solution: We have seen during the course that the differential operator in this exercise is symmetric and nonnegative. Thus it has only real eigenvalues, all eigenvalues are nonnegative and eigenfunctions corresponding to different eigenvalues are orthogonal.

$\lambda = 0$: The general solution to

$$-u'' = 0$$

is

$$u(x) = c_1 + c_2\, x.$$

The boundary conditions imply that

$$0 = u(0) = c_1, \quad 0 = u(\pi) = c_2\, \pi.$$

Thus $u \equiv 0$ is the only solution to $-u'' = 0$ satisfying the boundary conditions and 0 is not an eigenvalue.

$\lambda > 0$. The general solution to

$$-u'' = \lambda\, u$$

is

$$u(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x).$$

The boundary conditions yield that

$$0 = u(0) = c_1, \quad 0 = u(\pi) = c_2 \sin(\sqrt{\lambda}\, \pi).$$

A nontrivial solution exists if and only if $\sin(\sqrt{\lambda}\, \pi) = 0$. This is true if and only if $\sqrt{\lambda} = n\pi$ for some $n \in \mathbb{N}$.

Thus $\lambda_n = n^2$,$n = 1, 2, 3, \ldots$ are the eigenvalues of our differential operator and the preceding considerations show that

$$u_n(x) = \sin(nx)$$

is a basis for the eigenspace corresponding to the eigenvalue n^2.

b) Write the function $f(x) = x$ as

$$f = \sum_{n=1}^{\infty} c_n u_n,$$

u_n as in part a).

Solution:

$$f = \sum_{n=1}^{\infty} \frac{1}{\| u_n \|_2^2} \langle u_n, f \rangle u_n.$$

$$\| u_n \|_2^2 = \int_0^{\pi} |\sin(nx)|^2 dx = \pi/2.$$

.

$$\langle u_n, f \rangle = \int_0^{\pi} \sin(nx) \cdot x dx = -\frac{\pi}{n} (-1)^n.$$

Thus

$$f = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n-1} \sin(nx).$$

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7. The system

$$\begin{aligned} y_1'(t) &= \frac{1}{t} y_1(t) - y_2(t), \\ y_2'(t) &= \frac{1}{t^2} y_1(t) + \frac{2}{t} y_2(t), \end{aligned}$$

has the solution $x_1(t) = t^2$, $x_2(t) = -t$. Determine another, linearly independent, solution.

Hint: The ansatz $y(t) = \phi(t)x(t) + z(t)$ may be useful.

Solution: See the course book, W.Walter, ch. 15, V Example.

8. Determine the general solution to

$$\begin{aligned} y_1'(t) &= -4y_1(t) - 6y_2(t) + 3\sin(t) \\ y_2'(t) &= y_1(t) + y_2(t) + 2\sin(t). \end{aligned}$$

Solution: $\underline{y}' = A \cdot \underline{y} + \underline{b}$ where

$$A = \begin{pmatrix} -4 & -6 \\ 1 & 1 \end{pmatrix}, \quad \underline{b}(t) = \begin{pmatrix} 3\sin t \\ 2\sin t \end{pmatrix}.$$

$$\det(A-\lambda I) = \det\begin{pmatrix} -4-\lambda & -6 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2+3\lambda+2 = 0$$

if and only if

$$\lambda = \frac{-3 \pm \sqrt{9-4\cdot 1\cdot 2}}{2} = -1 \text{ eller } -2.$$

$A \cdot \binom{x}{y} = -\binom{x}{y} \Rightarrow x+y = -y \Rightarrow x = -2y$ and

$$\underline{v}_1 := \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ is an eigenvector to the eigenvalue } \lambda_1 := -1.$$

$A \cdot \binom{x}{y} = -2\binom{x}{y} \Rightarrow x+y = -2y \Rightarrow x = -3y$ and

$$\underline{v}_2 := \begin{pmatrix} -3 \\ 1 \end{pmatrix} \text{ is an eigenvector to the eigenvalue } \lambda_2 := -2.$$

$\underline{y}_1(t) := e^{\lambda_1 t} \cdot \underline{v}_1 = e^{-t} \cdot \binom{-2}{1}$ and

$\underline{y}_2(t) := e^{\lambda_2 t} \cdot \underline{v}_2 = e^{-2t} \cdot \binom{-3}{1}$ are solutions to

$$\underline{y} = A \cdot \underline{y}$$

and

$$Y(t) := \begin{pmatrix} -2e^{-t} & -3e^{-2t} \\ e^{-t} & e^{-2t} \end{pmatrix}$$

a fundamental matrix.

$$Y(t)^{-1} = \begin{pmatrix} e^{t} & 3e^{t} \\ -e^{2t} & -2e^{2t} \end{pmatrix},$$

$$Y(t)^{-1}\underline{b}(t) = \begin{pmatrix} 9e^{t}\sin t \\ -7e^{2t}\sin t \end{pmatrix}$$

$$\int e^{t}\sin t dt = \Big(\frac{1}{2}\sin t - \frac{1}{2}\cos t\Big)e^{t}$$

$$\int e^{2t}\sin t dt = \Big(\frac{2}{5}\sin t - \frac{1}{5}\cos t\Big)e^{2t}$$

$$Y(t)\int Y(s)^{-1}\underline{b}(s)ds = \begin{pmatrix} -9\sin t + 9\cos t + \frac{42}{5}\sin t - \frac{21}{5}\cos t \\ \frac{9}{2}\sin t - \frac{9}{2}\cos t - \frac{14}{5}\sin t + \frac{7}{5}\cos t \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5}\sin t + \frac{24}{5}\cos t \\ \frac{17}{10}\sin t - \frac{31}{10}\cos t \end{pmatrix} =: \underline{y}_p(t)$$

$$\underline{y}(t) = \underline{y}_p(t) + c_1\underline{y}_1(t) + c_2\underline{y}_2(t)$$

is the general solution to

$$\underline{y}' = A \cdot \underline{y} + \underline{b};$$

c_1, c_2 are arbitrary real numbers.

Good luck! Johannes