Recommended exercises for MAN460

(most of these exersices are taken from the book by Walter, some from Andersson, Böiers, and some from other sources)

1. Determine all solutions to the following equations, and determine the one passing through the origin

   (a)
   $$y' = \frac{y+1}{x+2} + \exp(\frac{y+1}{x+2})$$

   (b)
   $$y' = \frac{x+2y+1}{2x+y+2}$$

2. Determine all solutions to the following equations, and sketch the solutions. Determine all initial points (x_0, y_0) such that uniqueness does not hold for solutions passing through that point.

   (a)
   $$y' = 3(\mathrm{sgn}(y))|y|^{2/3}$$

   (b)
   $$y' = \sqrt{y(1-y)} \qquad\qquad y \leq 1$$

3. Use Picard's method for solving the equation
   $$x' = 2t(x+1), \qquad\qquad x(0) = 0.$$

4. Show that the solution $x(t)$ to the initial value problem
   $$x' = \sin(t) + \log(1+x^2), \qquad\qquad x(0) = 0$$
   is defined on the entire real axis.

5. Show that if $F(t)$ is an invertible matrix, then
   $$\frac{d}{dt}F(t)^{-1} = -F(t)^{-1}\frac{dF(t)}{dt}F(t)^{-1}$$

6. (a) Let $L(t, D)$ be a differential operator of second order, and suppose that y_1 and y_2 constitute a basis for the solutions to $L(t, D)y = 0$. Prove that the fundamental solution is
   $$E(t, \tau) = \frac{y_2(t)y_1(\tau) - y_1(t)y_2(\tau)}{W(\tau)}$$
   where $W(\tau)$ is the Wronskian $y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau)$

   (b) Show that if $L(t, D)y = y'' + a(t)y$, then the Wronskian is constant

7. (a) Let $L(t, D)$ be a second order differential operator, and assume that we know a solution $\tilde{y}$ to the equation $L(t, D)y = 0$. Show that the ansatz $y(t) = \tilde{y}(t)w(t)$ in this equation leads to a diffferential equation of first order, with $w'(t)$ as unknown.

   (b) Verify that $y(t) = t$ is a solution to the equation
   $$y'' + \frac{1}{t^2}y' - \frac{1}{t^3}y = 0$$
   and determine all solutions using the result from (a).

8. Let $x(t)$ be the solution to the initial value problem

$$x' = t^2 + x^2, \qquad x(0) = 1.$$

Show that $x(t) \to \infty$ when $t \to t_1$ for some t_1 with $\pi/4 \le t_1 \le 1$.
Hint: prove that

$$\frac{1}{1-t} \le x(t) \le \tan(t + \pi/4)$$

9. (a) Use successive approximations to find the general solution to the system $x' = Ax$, where A is a given $n \times n$ matrix.
   (b) By the use of Cauchy's polygon method, show that

$$\lim_{n\to\infty} \left(I + \frac{1}{n}A\right)^n = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

10. Consider the equation

$$x'' + \sin x = 0$$
$$x(0) = 0, \qquad x'(0) = 1\,.$$

Try to estimate the change in $x(t)$ when the initial condition $x'(0) = 1$ is replaced by $x'(0) = 1 + \delta$.

11. Determine the dimenson of the space of solutions to

$$x^{(4)} + x''' + x'' + x' = 0\,,$$

which remain bouded in the limit $t \to \infty$.

12. Compute e^A by summing the power series, when

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ d) $A = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$

13. Show that $\|e^A\| \le e^{\|A\|}$.

14. Show that if $S^* = -S$, then the matrix e^{tS} is orthogonal for all t.

15. Assume that $\mathbf{v}$ is an eigen vector with eigenvalue λ to the matrix A. Show that $\mathbf{x}(t) = te^{\lambda t}\mathbf{v}$ is a solution to the system $\mathbf{x}' = A\mathbf{x} + e^{\lambda t}\mathbf{v}$.

16. Suppose that the homogeneous system $\mathbf{x}' = A\mathbf{x}$ is asymptotically stable. Show that if $\mathbf{b}(t)$ is bounded for $t \ge t_0$, then every solution to the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ is bounded for $t \ge t_0$.

17. Find all equilibrium points to the following systems. Sketch the phase portraits.
   (a) $x' = y(x^2 + 1), \qquad y' = 2xy^2$
   (b) $x' = y(x^2 + 1), \qquad y' = -x(x^2 + 1)$
   (c) $x' = e^y, \qquad y' = e^y \cos x$
   (d) $x' = y(1 + x^2 + y^2), \qquad y' = -2x(1 + x^2 + y^2)$

18. Sketch the phase portrait to the system

$$\begin{aligned} x' &= -y + x\frac{1-x^2-y^2}{x^2+y^2} \\ y' &= x + y\frac{1-x^2-y^2}{x^2+y^2} \end{aligned}$$

19. (a) Sketch the curves $r^2 = a\sin^2(2\theta)$, where r, θ are polar coordinates in the xy-plane.
    (b) Give a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with this phase portrait

20. Determine a Lyapunov function for the following systems
    (a) $x' = -2x + xy^3, \qquad y' = -x^2y^2 - y^3$
    (b) $x' = -2xy - 2y^2, \qquad y' = x^2 - y^2 + xy$
    (c) $x' = -3x^3 - y, \qquad y' = x^5 - 2y^3$

21. Show that the origin is a stable equilibrium point to the system $x' = -2xy, \qquad y' = x^2 - y^2$. Is it asymptotically stable?

22. Sketch the phase portrait to the equation $x'' + 3x' + 2x = 0$, and determine a Lyapunov function for the equilibrium point.

23. Consider the equation $y'' + h(y') + g(y) = 0$, where $h(0) = g(0) = 0$. Assume that $sg(s) > 0$ when $s \neq 0$ but close to 0, and assume the same to hold for h. Show that

$$E(y, y') = \frac{1}{2}{y'}^2 + \int_0^y g(s)\,ds$$

is a Lyapunov function, and that the origin $y = y = 0$ is an asymptotically stable equilibrium point.

24. Determine the characteristics of the equilibrium point $x(t) \equiv 0$ to the equation
$x''' + \sqrt{1 + 2x + 8x'} - 1 + \arctan 4x'' = 0$

25. The *van der Pol* equation is $x'' + \mu(x^2 - 1)x' + x = 0$. How does the stability of the origin depend on the value of μ?