MATEMATIK
Tentamen 2005–05–28 kl. 08.45–13.45

Solutions and answers to
MAN460/TMA013 Ordinary differential equations

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1. Solve the following differential equations:

a) $y' = (1-y^2)/(1-t^2)$

Solution: $y(t) \equiv \pm 1$ are two solutions. If $y(t) \neq \pm 1$, we can divide the equation by $1-y^2$ to obtain

$$\frac{y'}{1-y^2} = \frac{1}{1-t^2}$$

Using $1/(1-t^2) = \frac{1}{2}\left((1-t)^{-1} + (1+t)^{-1}\right)$, and similarly for y, we find

$$\left(\frac{1}{1-y(t)} + \frac{1}{1+y(t)}\right) y'(t) = \frac{1}{1-t} + \frac{1}{1+t}$$

which can be integrated to get

$$\log\left(\frac{1+y(t)}{1-y(t)}\right) = \log\left(\frac{1+y(0)}{1-y(0)}\right) + \log\left(\frac{1+t}{1-t}\right),$$

that is,

$$y(t) = \frac{(1+t)(1+y_0) - (1-t)(1-y_0)}{(1+t)(1+y_0) + (1-t)(1-y_0)}.$$

Note that this formula also includes the cases $y(t) \equiv \pm 1$.

b) $y' + y = \cos t$

Solution This is a linear equation, which has the solution

$$\begin{aligned} y(t) &= e^{-t} y_0 + e^{-t} \int_0^t e^{\tau} \cos \tau \, d\tau \\ &= e^{-t} y_0 + \frac{1}{2}\left(\cos t + \sin t - e^{-t}\right). \end{aligned}$$

2. Construct a Lyapunov function for the equation

$$x' = -2xy - 2y^2, \qquad y' = x^2 - y^3 + xy$$

Solution: Take for example $V(x,y) = x^2 + 2y^2$. Then $V(x,y)$ is positive, and zero only when $x = y = 0$, and

$$\dot{V}(x,y) = 2x(-2xy - 2y^2) + 4y(x^2 - y^3 + xy) = -4y^4 \leq 0.$$

3. Let A and B be two $n \times n$-matrices. In which cases is it true that $e^A e^B = e^{A+B}$? Give a proof for your answer.

   Solution: Theory from the book(s)

4. a) Compute $\exp(At)$, where $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

   Solution: The characteristic polynomial is $p(\lambda) = (1-\lambda)^2 - 9 = (\lambda - 4)(\lambda + 2)$. Then

$$e^{\lambda t} \quad = \quad g(\lambda)(\lambda-4)(\lambda+2) + a\lambda + b\,,$$

   for some analytic function $g(\lambda)$. Setting $\lambda = 4$ and $\lambda = -2$ in this equation gives $a = \frac{1}{6}\left(e^{4t} - e^{-2t}\right)$ and $b = \frac{1}{3}\left(e^{4t} + 2e^{-2t}\right)$, and therefore the Caley-Hamilton theorem

$$\begin{aligned} e^{At} &= \frac{1}{6}\left(e^{4t} - e^{-2t}\right) A + \frac{1}{3}\left(e^{4t} + 2e^{-2t}\right) I \\ &= \frac{1}{2} e^{4t} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} e^{-2t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

   Remark: This could have been obtained as easily by diagonalization of the matrix A.

   b) Compute $(1 + \frac{1}{n} tA)^n$, where A is the same matrix as above. Compare the two results.

   Solution: As before,

$$\left(1 + \frac{1}{n} t\lambda\right)^n \quad = \quad g(\lambda)(\lambda-4)(\lambda+2) + a\lambda + b\,,$$

   for some (other) analytic function g, and new constants a and b. Plugging in $\lambda = 4$ and $\lambda = -2$ gives

$$\begin{aligned} a &= \frac{1}{6}\left(\left(1 + \frac{4t}{n}\right)^n + \left(1 - \frac{2t}{n}\right)^n\right), \\ b &= \frac{1}{3}\left(\left(1 + \frac{4t}{n}\right)^n - 2\left(1 - \frac{2t}{n}\right)^n\right). \end{aligned}$$

   Hence

$$\left(1 + \frac{1}{n} tA\right)^n \quad = \quad \frac{1}{2}\left(1 + \frac{4t}{n}\right)^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\left(1 - \frac{2t}{n}\right)^n \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

   This gives an approximation to the result in a, because $(1 + \alpha/n)^n \to e^\alpha$ when $n \to \infty$.

5. Show that if L is a symmetric differential operator of order n, then n must be an even number.

Solution: Let $Lf(t) = a_n(t)f^{(n)}(t) + a_{n-1}(t)f^{(n-1)}(t) + \dots a_0(t)f(t)$. Since the inner product is not specified, we choose the most natural one, $(f, g) = \int_I f(t)g(t)\,dt$, where I is some interval. Then for a function g which is zero at the boundaries of I,

$$(Lf, g) = \int_I a_n(t)f^{(n)}(t)g(t)\,dt + \quad \text{terms with lower order derivatives.}$$

By partial integration n times, we get

$$\begin{aligned}
(Lf, g) &= -\int f^{(n-1)}(t)\frac{d}{dt}\left(a_n(t)g(t)\right)\,dt + \quad \text{terms with lower order derivatives} \\
& \dots \\
&= (-1)^n \int f(t)\frac{d^n}{dt^n}\left(a_n(t)g(t)\right)\,dt + \quad \text{terms with lower order derivatives} \\
&= (-1)^n \int f(t)a_n(t)g^{(n)}(t)\,dt + \quad \text{terms with lower order derivatives} \\
&= (-1)^n \int f(t)Lg(t)\,dt + \quad \text{terms with lower order derivatives.}
\end{aligned}$$

Hence, to have $(Lf, g) = (f, Lg)$, we need that $(-1)^n = 1$, and that all the terms "with lower order derivatives" vanish.

6. a) Define a Green's function for a boundary value problem of the Sturm-Liouville type. State fundamental properties of this function.

   Solution: Theory from the book(s)

   b) Construct the Green's function explicitly for the problem

$$\begin{aligned}
-u''(x) + u(x) &= 0 \\
u(0) &= 0 \\
u'(\pi/2) - u(\pi/2) &= 0
\end{aligned}$$

   Solution: Two independent solutions to $-u''(x) + u(x) = 0$ can be found as linear combinations of e^x and e^{-x}. To find a function that satisfies the boundary condition at $x = 0$, take $u_l(x) = e^x - e^{-x}$, and one that satisfies the boundary condition at $x = \pi/2$ is $u_r(x) = e^x$. Referring to the standard notation in the Sturm-liouville theory $p = -1$, and then $c = -(u_l(x)u_r'(x) - u_l'(x)u_r(x)) = 2$, and therefore we find the Green's function as

$$\Gamma(x, \xi) = \frac{1}{2}\begin{cases} \left(e^{\xi} - e^{-\xi}\right)e^{x} & (\xi \le x \le \pi/2) \\ \left(e^{x} - e^{-x}\right)e^{\xi} & (0 \le x \le \xi) \end{cases}$$

7. State and prove a local existence theorem for systems of first order equations. How can this theorem be used for proving existence and uniqueness to euqations of higher order?

   Solution: Theory from the book(s)

8. Consider the equation

$$\mathbf{y}' = A(t)\mathbf{y}$$

   where

$$A(t) = \begin{pmatrix} 1 & t & t^2 \\ t & t^2 & 2 \\ 1 & t^3 & 1 \end{pmatrix}$$

   Prove that the set of solutions to this equation is a three dimensional linear space.

   Solution: Theory from the book(s), because it is a linear differential equation.

Good luck!

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