Kripke Semantics

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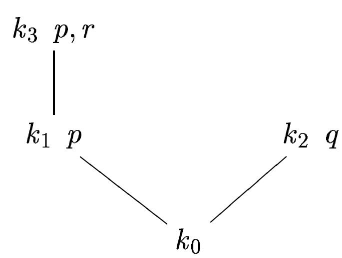
For constructive logic it is not possible to have a semantics of just two truth values. Instead we will here introduce a semantics which uses partial orders: the nodes of the ordering can be seen as stages of knowledge.

The formal definition is as follows.

> A *Kripke model* consists of a non-empty partial order $\leq$ and a monotone assignment of propositional variables to the nodes of the ordering.

The assignment of a propositional variable p to a node means intuitively that we know at that stage that p holds. That the assignment is monotone means that once we know that a proposition is true, we also know that it is true at later stages. We only require the ordering to be partial since at a given stage there may be different ways to extend the knowledge.

Here is an example:



At the root node, k_0, no atomic truth is known and there are two ways to proceed: to k_1 where p is known, or to k_2 where q is known. From k_2 there is no possibility to extend our knowledge, but from k_1 we may proceed to k_3 where we get to know r.

We will now define what it means for a propositional formula A to be true at a node k, which we write $k \Vdash A$ and say that k *forces* A. The definition is by recursion on the construction of the formula A.

- $k \Vdash p$ if the propositional variable p is assigned to the node k.
- $k \Vdash B \wedge C$ if $k \Vdash B$ and $k \Vdash C$.
- $k \Vdash B \vee C$ if $k \Vdash B$ or $k \Vdash C$.

- $k \Vdash B \to C$ if for all $l \geq k$, if $l \Vdash B$ then $l \Vdash C$.
- $\bot$ is not forced at any node.

In the example above, neither any disjunction nor any conjunction is forced at k_0, but $k_0 \Vdash r \to p$.

Since $\neg A$ is defined to be $A \to \bot$, we see that $\neg A$ is forced at a node k if and only if A is not forced at any node greater than or equal to k. So, in the example above, $k_1 \Vdash \neg q$, $k_2 \Vdash \neg p$ and $k_2 \Vdash \neg r$, but none of the negations of p, q and r are forced at k_0.

The next proposition tells us that if a formula is forced at a node, it is also forced at all greater nodes.

Proposition 1 (Monotonicity) *Let k be a node in a Kripke model and A a formula such that $k \Vdash A$. If $l \geq k$, then $l \Vdash A$.*

Proof. Induction on the construction of the formula A. Let l be an arbitrary node such that $l \geq k$.

1. A is a propositional variable p. By the definition of a Kripke model, the assignment of propositional variables to the nodes must be monotone; hence, $l \Vdash p$.

2. A is $B \wedge C$. That $k \Vdash B \wedge C$ means that $k \Vdash B$ and $k \Vdash C$. By induction hypothesis we know that $l \Vdash B$ and $l \Vdash C$; hence, $l \Vdash B \wedge C$.

3. A is $B \vee C$. That $k \Vdash B \vee C$ means that $k \Vdash B$ or $k \Vdash C$. If $k \Vdash B$, the induction hypothesis gives that $l \Vdash B$; hence $l \Vdash B \vee C$. The case that $k \Vdash C$ is handled in the same way.

4. A is $B \to C$. That $k \Vdash B \to C$ means that for all $l' \geq k$, if $l' \Vdash B$ then $l' \Vdash C$. Let $l'' \geq l$. Transitivity of $\leq$ gives $l'' \geq k$; hence, since $k \Vdash B \to C$, if $l'' \Vdash B$ then $l'' \Vdash C$ as desired.

We let $\vdash$ denote derivability in intuitionistic propositional logic, that is, the usual rules except RAA.

Proposition 2 *Let $\Gamma \vdash A$. If all formulas in Γ are forced at a node in a Kripke model then also A is forced at that node.*

Proof. Let a Kripke model be given. We use induction on the derivation $\Gamma \vdash A$ to show that if all formulas in Γ are forced at a node in the model then also A is forced at that node. We use

$$\begin{array}{c}\Delta \\ D\end{array}$$

to denote a derivation of the formula D from the set Δ. We will only treat a few of the rules.

1. If A is in Γ, the conclusion is trivial.

2. A is $B \wedge C$ and is obtained by $\wedge$-introduction,

$$\frac{\begin{array}{cc}\Gamma_1 & \Gamma_2 \\ B & C\end{array}}{B \wedge C}$$

   where $\Gamma_1 \cup \Gamma_2 = \Gamma$. Let k be an arbitrary node of the model and assume that all formulas in Γ are forced at k. Since $\Gamma_1 \subset \Gamma$ and $\Gamma_2 \subset \Gamma$, the induction hypothesis directly gives that both B and C are forced at k; hence, $B \wedge C$ is forced at k.

3. A is $B \to C$ and is obtained by $\to$-introduction,

$$\frac{\begin{array}{c}\Gamma \cup \{B\} \\ C\end{array}}{B \to C}$$

   Let k be an arbitrary node at which all formulas of Γ are forced. We must show that for any node $l \geq k$, if $l \Vdash B$ then $l \Vdash C$. By proposition 1, we know that all formulas of Γ are forced at l. The induction hypothesis tells us that C is forced at all nodes which forces all formulas of Γ and B; hence, $B \to C$ is forced at k.

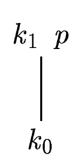
By putting Γ equal to the empty set in proposition 2, we obtain

Corollary 1 (Soundness) *If $\vdash A$, then A is forced at all nodes in every Kripke model.*

Examples

Soundness makes it possible to use Kripke models to show that certain formulas cannot be proved in intuitionistic propositional logic.

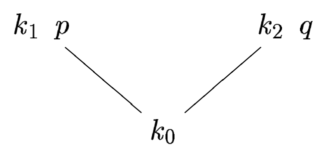
Example 1 In the Kripke model



p is not forced at k_0, neither is $p \to \bot$ since p is forced at k_1 and $k_0 \leq k_1$; so $p \vee \neg p$ is not forced at k_0. Hence, soundness gives that the law of the excluded middle cannot be proved without *reductio ad absurdum.*

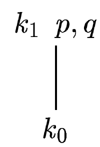
We can also use this model to show that $\neg\neg p \to p$ cannot be derived in intuitionistic logic: we just showed that $\neg p$ is not forced at any node; hence $\neg\neg p$ is forced at k_0. Since p is not forced at k_0, $\neg\neg p \to p$ is not forced at k_0.

Example 2 In the model



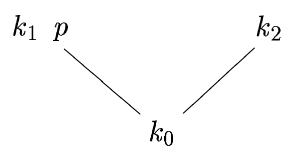
neither $p \to q$ nor $q \to p$ is forced at k_0; hence $(q \to p) \vee (q \to p)$ is not forced at k_0.

Example 3



At k_0, $p \to q$ is forced but not $\neg p \vee q$. So $(p \to q) \to (\neg p \vee q)$ cannot be proved without *RAA*.

Example 4 $\neg\neg p \to p$ is forced at the bottom node in



but not $p \vee \neg p$; this shows that $(\neg\neg p \to p) \to (p \vee \neg p)$ does not hold constructively.

Exercises

1. Show that $\neg\neg p \to (p \vee \neg p)$ is not forced at the bottom node of the model in Example 1.

2. Construct a counter model to

   (a) $((p \to q) \to p) \to p$ (Peirce's law)

   (b) $\neg\neg p \vee \neg p$.

   (c) $(p \to (q \vee r)) \to ((p \to q) \vee (p \to r))$.

   (d) $\neg(p \wedge q) \to \neg p \vee \neg q$.

3. Fill in the details for the remaining rules in the proof of proposition 2.