Suggested solutions
Mathematical Logic 2002-01-18

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1. a) Suppose $\varphi, \psi \vDash \sigma$ then for every model $\mathfrak{M}$, if $\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \psi$ then $\mathfrak{M} \vDash \sigma$. Let $\mathfrak{M}$ be an arbitrary model, if φ and ψ holds in $\mathfrak{M}$ then by the assumption, σ also holds in $\mathfrak{M}$, so $\mathfrak{M} \vDash (\varphi \wedge \psi) \to \sigma$, so $\vDash (\varphi \wedge \psi) \to \sigma$.

On the other hand if $\vDash (\varphi \wedge \psi) \to \sigma$ then $\mathfrak{M} \vDash (\varphi \wedge \psi) \to \sigma$ for any model $\mathfrak{M}$, so if $\mathfrak{M}$ is a model in which φ and ψ is true then also σ is true since $\mathfrak{M} \vDash (\varphi \wedge \psi) \to \sigma$, so $\varphi, \psi \vDash \sigma$.

b) By Lemma 1.5.4 in van Dalen it is enough to find a valuation v such that $v(p_{2k}) = v(\neg p_{2k+1}) = 1$ for all natural numbers k. This is easy, let $v(p_{2k}) = 1$ and $v(p_{2k+1}) = 0$ for every natural number k.

2. a)

$$\dfrac{\dfrac{\dfrac{[\neg p]_2 \quad [\neg\neg p]_1}{\bot} \to \mathrm{E}}{p}\ \mathrm{RAA}_2}{\neg\neg p \to p} \to \mathrm{I}_1$$

b)

$$\dfrac{\dfrac{[p \vee \bot]_1 \quad [p]_2 \quad \dfrac{[\bot]_3}{p}\ \bot\mathrm{E}}{p}\ \vee\mathrm{E}_{2,3}}{(p \vee \bot) \to p} \to \mathrm{I}_1$$

c)

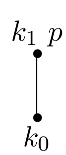
$$\dfrac{\dfrac{\dfrac{[\varphi(a)]_1 \quad \dfrac{\forall x(x = a)}{x = a}\ \forall\mathrm{E}}{\varphi(x)}\ \mathrm{RI4}}{\forall x \varphi(x)}\ \forall\mathrm{I}}{\varphi(a) \to \forall x \varphi(x)} \to \mathrm{I}_1$$

3. $(i) \Rightarrow (ii)$ Suppose there is a φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, then by an application of $\to$ E we get $\Gamma \vdash \bot$, i.e., Γ is not consistent, contradicting (i).

$(ii) \Rightarrow (iii)$ Since $\Gamma \vdash \psi \to \psi$, (ii) gives us $\Gamma \nvdash \neg(\psi \to \psi)$, so there is a formula which can not be proved.

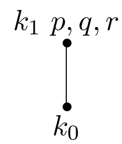
$(iii) \Rightarrow (i)$ If Γ is not consistent, i.e., $\Gamma \vdash \bot$ then by an application of $\bot$E we get $\Gamma \vdash \varphi$ for any formula φ, contradicting (iii).

4. a) Let $\mathfrak{K}$ be the Kripke model



then $k_1 \Vdash p$ so $k_0 \nVdash \neg p$ and therefore $k_0 \Vdash \neg\neg p$ (since also $k_1 \nVdash \neg p$) and so $k_0 \nVdash \neg\neg p \to p$.

b) Let $\mathfrak{K}$ be the Kripke model



then $k_1 \Vdash p$, so $k_0 \nVdash \neg p$ and also $k_0 \nVdash \neg q$ and $k_0 \nVdash \neg r$. This implies that $k_0 \nVdash \neg p \vee \neg q \vee \neg r$ and we also have that $k_1 \nVdash \neg p \vee \neg q \vee \neg r$, so $k_0 \Vdash \neg(\neg p \vee \neg q \vee \neg r)$. This ends up to that $k_0 \nVdash \neg(\neg p \vee \neg q \vee \neg r) \rightarrow (p \wedge q \wedge r)$, since $k_0 \nVdash p \wedge q \wedge r$.

5. a) Let $\mathfrak{M} = \langle \{0,1\}, \{\langle 0,0\rangle, \langle 1,1\rangle\}\rangle$, i.e., the interpretation of P is true only for the pairs $\langle 0,0\rangle$ and $\langle 1,1\rangle$. Then it is clear that $\mathfrak{M} \models \forall x \exists y P(x,y)$ since we can choose $y = x$, but $\mathfrak{M} \not\models \exists y \forall x P(x,y)$ since for every value of y there is a value of x ($x = 1 - y$) such that $P(x,y)$ does not hold. This implies that

$$\mathfrak{M} \not\models \forall x \exists y P(x,y) \rightarrow \exists y \forall x P(x,y).$$

b) Let $\mathfrak{M} = \langle \{0,1\}, \{0\}, F\rangle$, i.e., the interpretation of Q is true only for 0 and the interpretation of R is False. Then $\mathfrak{M} \models \forall x Q(x) \rightarrow R$ but $\mathfrak{M} \not\models \forall x (Q(x) \rightarrow R)$ since for $x = 0$, Q is true and R false. This implies that

$$\mathfrak{M} \not\models (\forall x Q(x) \rightarrow R) \rightarrow \forall x (Q(x) \rightarrow R).$$

6. a) Let us work in the language with two 0-ary predicate symbols P and Q and let

$$\begin{aligned} T_1 &= \{\varphi : P \rightarrow Q \vdash \varphi\} \text{ and} \\ T_2 &= \{\varphi : P \vdash \varphi\}. \end{aligned}$$

Both T_1 and T_2 are theories (easy to check), I claim that $Q \notin T_1 \cup T_2$ which proves that $T_1 \cup T_2$ is not a theory since $T_1 \cup T_2 \vdash Q$ (follows from the fact that $P \rightarrow Q, P \vdash Q$). To see that $Q \notin T_i$ for $i = 1, 2$ we can construct a model of $T_i \cup \{\neg Q\}$, which shows that $T_i \nvdash Q$. These models are easily constructed (in the first case let P and Q be false and in the second let P be true and Q false).

b) If not, let

$$\Gamma' = \Gamma \cup \{\lambda_k : k \in \mathbb{N}\},$$

where λ_k expresses "there are at least k elements" (see p. 83 in van Dalen). By the compactness theorem Γ' is consistent since every finite subset of Γ' is contained in one of

$$\Gamma_n = \Gamma \cup \{\lambda_k : k \leq n\},$$

and Γ has arbitrary large models, so for every n there is a model of Γ_n. By the model existence lemma there is a model of Γ', but this model is an infinite model of Γ, a contradiction.

7.

$$\begin{gathered} (\forall x Q(x) \rightarrow \exists x Q(x)) \wedge P(x,y) \\ (\forall z Q(z) \rightarrow \exists w Q(w)) \wedge P(x,y) \\ \exists z \exists w (Q(z) \rightarrow Q(w)) \wedge P(x,y) \\ \exists z \exists w ((Q(z) \rightarrow Q(w)) \wedge P(x,y)) \end{gathered}$$

All the formulas above are equivalent and the last one is in prenex form.

8. See the extra material on Gödel's incompleteness theorem.