MATHEMATICS
Chalmers and Göteborg University

Date: 3 January 2005
Material allowed: only the three attached pages with formulas
Telephone: 076-2186654
Time: 8.30 – 13.30

Exam in Fourier Analysis
MAN 530, TMA362

1. State the theorem about termwise differentiation of Fourier series, for Fourier series in both real and complex form.

2. Determine the eigenvalues and eigenfunctions of the Sturm-Liouville problem
$$f'' + \lambda f = 0, \qquad f(0) = 0, \qquad f'(\ell) = 0.$$
Use these eigenfunctions to find the solution in series form of the initial value problem
$$\begin{aligned} u_t &= k u_{xx}, \qquad 0 < x < \ell,\ \ t > 0 \\ u(0,t) &= 0, \quad u_x(\ell, t) = 0 \\ u(x,0) &= g(x). \end{aligned}$$
Here $k > 0$, and g is a given function.

3. Determine the Fourier series of the function $\cosh x$ in the interval $[-\pi, \pi]$, in real form.

4. Let $h(x) = 1$ for $0 \le x < \pi/2$ and $h(x) = 0$ for $\pi/2 \le x \le \pi$. Determine those constants a and b which make the integral
$$\int_0^{\pi} |h(x) - a\cos x - b\sin x|^2 \, dx$$
as small as possible.

5. Compute for $a > 0$
$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+4)^2}\, dx,$$
for instance by means of Fourier transforms.

6. Solve the initial value problem
$$\begin{aligned} u_t(x,t) &= 2u_{xx}(x,t), \qquad 0 < x < \pi/2, \quad t > 0 \\ u(0,t) &= 1, \quad u(\pi/2, t) = -1 \\ u(x,0) &= 0. \end{aligned}$$

Solutions to
exam in Fourier Analysis

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2.

(Short comments only.) The eigenvalues are $((n-1/2)\pi/\ell)^2, \quad n = 1, 2, ...,$ and the normalised eigenfunctions are $\sqrt{2/\ell}\sin(n-1/2)\pi x/\ell$. One has

$$u(x,t) = \sum_1^\infty b_n \exp\left(-((n-1/2)\pi/\ell)^2 kt\right) \sin(n-1/2)\pi x/\ell,$$

where

$$b_n = \frac{2}{\ell}\int_0^\ell g(x)\sin(n-1/2)\pi x/\ell\, dx.$$

3.

The 2π-periodic function f which coincides with the cosh function in $[-\pi, \pi]$ is piecewise smooth, continuous and even. Its Fourier series is therefore a cosine series

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx.$$

The theorem about termwise differentiation of Fourier series applies and says that the Fourier series of $f'(x)$ is

$$-\sum_1^\infty n a_n \sin nx.$$

But since $f'(x) = \sinh x$ for $-\pi < x < \pi$, this series must be the one given by entry 20 of the Fourier series table. Thus

$$-na_n = \frac{2\sinh\pi}{\pi}\frac{(-1)^{n+1}n}{n^2+1}, \quad n = 1, 2, ...$$

For $n = 0$ we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cosh x\, dx = \frac{2\sinh\pi}{\pi}.$$

The required series is then

$$\cosh x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{0}^{\infty} \frac{(-1)^n}{n^2+1} \cos nx.$$

We remark that there are several other methods. It is possible to consider the cosh function as the derivative of the sinh function, but then one must apply the theorem about termwise *integration* of Fourier series. It is also possible to use the Fourier seris of $e^{\pm x}$ found in entry 18 of the table. Of course one can instead compute the desired Fourier coefficients directly by integrating.

4.
One verifies that $\cos x$ and $\sin x$ are orthogonal in $[0, \pi]$. The corresponding normalised functions, with normalisation in $L^2([0, \pi])$, are

$$\phi(x) = \sqrt{\frac{2}{\pi}} \cos x \qquad \text{and} \qquad \psi(x) = \sqrt{\frac{2}{\pi}} \sin x$$

The theorem about best approximation then tells us that the best approximation of h in $L^2([0, \pi])$ is

$$a' \phi(x) + b' \psi(x),$$

where

$$a' = \int_0^\pi h(x) \phi(x)\, dx \qquad \text{and} \qquad b' = \int_0^\pi h(x) \psi(x)\, dx.$$

One finds that $a' = b' = \sqrt{2/\pi}$, which implies that $a = b = 2/\pi$.

5.
The integrand can be written as a product

$$f(x) = \frac{1}{x^2+4} \cdot \frac{1}{2} \frac{e^{iax} + e^{-iax}}{x^2+4}.$$

From the table of Fourier transforms, we see that its Fourier transform is given by a convolution; indeed

$$\hat{f}(\xi) = \frac{1}{2\pi} \frac{\pi}{2} e^{-2|\xi|} * \frac{1}{2} \frac{\pi}{2} \left(e^{-2|\xi - a|} + e^{-2|\xi + a|} \right).$$

Since $\int_{-\infty}^{\infty} f(x)\,dx = \hat{f}(0)$, we get by evaluating this convolution at 0

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{16}\int_{-\infty}^{\infty} \left(e^{-2|\xi|-2|\xi-a|} + e^{-2|\xi|-2|\xi+a|}\right)\,d\xi.$$

Since the integrand is a sum of two terms, we can obviously write the integral as a sum of two integrals. Making in one of these integrals the transformation $\xi \mapsto -\xi$, we get the other, so the two integrals are equal. Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \frac{\pi}{8}\int_{-\infty}^{\infty} e^{-2|\xi|-2|\xi-a|}\,d\xi \\
&= \frac{\pi}{8}\left(\int_{-\infty}^{0} e^{2\xi-2(a-\xi)}\,d\xi + \int_{0}^{a} e^{-2\xi-2(a-\xi)}\,d\xi + \int_{a}^{\infty} e^{-2\xi-2(\xi-a)}\,d\xi\right) \\
&= \frac{\pi}{8}\left(\frac{1}{4}e^{-2a} + ae^{-2a} + \frac{1}{4}e^{-2a}\right) \\
&= \frac{\pi}{16}(1+2a)e^{-2a}.
\end{aligned}$$

6.

We first determine the steady state solution $u_0(x)$. Since u_0 must satisfy the heat equation, it is an affine function, $u_0(x) = ax + b$. Since it must also satisfy the boundary conditions at $x = 0$ and $x = \pi/2$, one finds that $u_0(x) = 1 - 4x/\pi$.

The difference $v(x,t) = u(x,t) - u_0(x)$ is a solution of the heat equation with boundary values 0 and initial values $v(x,0) = 4x/\pi - 1$. So for v we have a standard problem and the solution is given by

$$v(x,t) = \sum_{1}^{\infty} b_n e^{-2(2n)^2 t} \sin 2nx.$$

Here the b_n are the coefficients of the boundary values $4x/\pi - 1$, expanded in a sine series in $[0, \pi/2]$. Letting $\theta = 2x$ in entries 1 and 6 in the Fourier series table, we get that

$$x = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin 2nx$$

and

$$1 = \frac{4}{\pi}\sum_{1}^{\infty} \frac{1}{2n-1} \sin 2(2n-1)x = \frac{4}{\pi}\sum_{m \text{ odd}} \frac{1}{m} \sin 2mx$$

for $0 < x < \pi/2$. So we get $b_n = 0$ for odd n and $b_n = -4/(\pi n)$ for even n. Altogether then

$$u(x,t) = 1 - \frac{4x}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-32k^2 t}\sin 4kx.$$

Instead of table entries 1 and 6, one can use entry 3.