Exercises in Fourier Analysis 2003

1. Assume that the function $f(x)$ is 2-periodic, and $f(x) = (x+1)^2$ for $-1 < x < 1$. Expand $f(x)$ in complex trigonometric Fourier series. Find a 2-periodic solution to the equation
$$2y'' - y' - y = f(x).$$

2. The function $f(t)$ is 3-periodic, and
$$f(t) = \begin{cases} t & \text{if} \quad 0 \le t \le 1, \\ 1 & \text{if} \quad 1 < t < 2, \\ 3-t & \text{if} \quad 2 \le t \le 3. \end{cases}$$
Find, in the form trigonometric Fourier series, a periodic solution to the differential equation
$$y'' + 3y = f(t).$$

3. Expand the function $\cos x$ in sine-series on the interval $(0, \frac{\pi}{2})$. Use the result to calculate
$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}.$$

4. Let $f(t) = 1 - t^2$ for $|t| \le 1$ and f be 2-periodic. Find a bounded solution to
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & t > 0, -\infty < x < \infty, \\ u(0,t) = f(t), & -\infty < t < \infty. \end{cases}$$

5. Solve the Laplace equation $\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$ on the annulus $1 < r < 2$ (in polar coordinates) with boundary condition $u(1,\theta) = 0, u(2,\theta) = f(\theta)$, where $f(\theta)$ is 2π-periodic, and
$$f(\theta) = 1 - \frac{\theta^2}{\pi^2} \quad \text{for} \quad |\theta| \le \pi.$$

6. Find the Fourier transform of the function
a) $\dfrac{t}{(t^2+a^2)^2}$, b) $\dfrac{1}{(t^2+a^2)^2}$, c) $\dfrac{t}{(t^2+1)(t^2+2t+5)}$,
d) $e^{-a|t|}\sin bt \quad (a > 0, b > 0)$.

7. The function $f(t)$ has Fourier transform $\hat{f}(\omega) = \frac{\omega}{1+\omega^4}$. Find
a) $\displaystyle\int_{-\infty}^{\infty} tf(t)dt$, b) $f'(0)$.

8. The function $f(t)$ has Fourier transform $\dfrac{1-i\omega}{1+i\omega}\dfrac{\sin\omega}{\omega}$. Find $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt$.

9. Find $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}dx$ by using Fourier transform.

10. The function $f(t)$ has Fourier transform $\dfrac{1}{|\omega|^3+1}$. Find $\displaystyle\int_{-\infty}^{\infty} |f * f'|^2 dt$, where $*$ denotes convolution.

11. Compute the Fourier transform of the function

$$f(t) = \int_0^2 \frac{\sqrt{\omega}}{1+\omega} e^{i\omega t} d\omega.$$

Find also a) $\displaystyle\int_{-\infty}^{\infty} f(t) \cos t dt$, b) $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt$.

12. Let $f(t) = \displaystyle\int_0^1 \sqrt{\omega} e^{\omega^2} \cos \omega t d\omega$. Find $\displaystyle\int_{-\infty}^{\infty} |f'(t)|^2 dt$.

13. Find en solution to the equation

$$u'(t) + 2u(t) + e^{-2t} \int_{-\infty}^{t} e^{2\tau} u(\tau) d\tau = \delta(t).$$

14. Solve the integral equation

$$\int_0^{\infty} e^{-\tau} u(t-\tau) d\tau - \int_{-\infty}^{0} e^{\tau} u(t-\tau) d\tau = \sqrt{3} u(t) - e^{-|t|}.$$

15. Find a solution to the equation

$$u(t) + \int_{-\infty}^{t} e^{\tau - t} u(\tau) d\tau = e^{-2|t|}.$$

16. For certain linear, time-invariant system the input signal $\dfrac{1}{1+t^2}$ gives output-signal $\dfrac{t}{(4+t^2)^2}$. Find impulse response, in $\cos \omega t$. Is the system causal, stable?

17. A linear, time-invariant system has impulse response $h(t) = e^{-4t^2}$. Let $y(t)$ be the response to the input signal e^{-t^2}. Find $\displaystyle\int_{-\infty}^{\infty} e^{it} h(t) y(t) dt$.

18. For certain linear, time-invariant system the input signal $\dfrac{1}{4+t^2}$ give out-signal e^{-2t^2}. Find the out-signal (in the form of a complex Fourier series), if the input-signal is

$$\sum_{n=-\infty}^{\infty} [2\delta(t-2n) - \delta(t-2n-1)].$$

19. Let $x(n)$ be N-periodic, and

$$x(n) = \begin{cases} 1, & \text{då} \quad 0 \le n \le k-1, \\ 0, & \text{då} \quad k \le n \le N-1. \end{cases}$$

Find the discrete Fourier transform and use Parseval formula to calculate

$$\sum_{\mu=1}^{N-1} \frac{1 - \cos \frac{2\pi\mu k}{N}}{1 - \cos \frac{2\pi\mu}{N}}.$$

20. Find the discrete Fourier transform of the signal (sequence) $x(n) = \sin \dfrac{n\pi}{N}$, $\quad n = 0, \ldots, N-1$, and $x(n)$ is N-periodic.

21. Prove att functions $\varphi_n(x) = \dfrac{\sin\frac{x}{2}}{\pi x} e^{inx}$ are pair wise orthogonal in $L^2(\mathbf{R})$. Find the number c_n so that

$$\int_{-\infty}^{\infty} \left| \frac{1}{1+x^2} - \sum_{n=-N}^{N} c_n \varphi_n(x) \right|^2 dx$$

is minimized.

22. Find den solution $y(x)$ to $y'' - y = 0$ that minimizes $\displaystyle\int_{-1}^{1} [1 + x - y(x)]^2 dx$.

23. Find the all egenvalues and egenfunctions to Sturm-Liouville problem

$$\begin{cases} f'' + \lambda f = 0, & 0 < x < a, \\ f(0) - f'(0) = 0, & f(a) + 2f'(a) = 0. \end{cases}$$

24. Find the all eigenvalues and eigenfunctions to Sturm-Liouville problem

$$\begin{cases} -e^{-4x} \dfrac{d}{dx}\left(e^{4x} \dfrac{du}{dx}\right) = \lambda u, & 0 < x < 1, \\ u(0) = 0, \quad u'(1) = 0. \end{cases}$$

Expand the function e^{-2x} in Fourier series in terms of the eigenfunctions.

25. Solve the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = y, & 0 < x < 2,\ 0 < y < 1, \\ u(x,0) = 0, & u(x,1) = 0, \\ u(0,y) = y - y^3, & u(2,y) = 0. \end{cases}$$

26. Solve the problem

$$\begin{cases} \sqrt{1+t}\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, & 0 < x < 1,\ t > 0, \\ u(0,t) = 1, & u(1,t) = 0, \\ u(x,0) = 1 - x^2. \end{cases}$$

27. Solve the problem

$$\begin{cases} u''_{xx} + u''_{yy} + 20u = 0, & 0 < x < 1,\ 0 < y < 1, \\ u(0,y) = u(1,y) = 0, & \\ u(x,0) = 0, & u(x,1) = x^2 - x. \end{cases}$$

28. Solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = t \sin x, & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, & \\ u(x,0) = \sin 2\pi x. \end{cases}$$

29. Solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1,\, t > 0, \\ u(0,t) = t+1, & \\ u(1,t) = 0, & \\ u(x,0) = 1-x. & \end{cases}$$

30. Solve the Laplace equation $\Delta u = 0$ on the region $0 < \theta < \dfrac{\pi}{4}$, $1 < r < 2$, (in polar coordinates on planet) with boundary condition

$$\begin{cases} u = 0 \text{ for } r = 1, \quad u'_r = 0 \text{ for } r = 2, \\ u = 0 \text{ for } \theta = 0, \quad u = r - 1 \text{ for } \theta\dfrac{\pi}{4}. \end{cases}$$

31. Expand the function $\sin(2\sin x)$ in trigonometric Fourier series (real form).

32. A periodic external force $q \sin \omega t$ acts uniformly on a circular membrane with radius a The transversal oscillation $u(r,t)$ satisfies the equation

$$\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{q}{S}\sin \omega t, \quad u|_{r=a} = 0.$$

Find den stationary oscillating motion (namely a solution of form $u(r,t) = v(r)\sin \omega t$). What are the resonance (angel) frequencies?

33. Solve the heat equation $u'_t = \Delta u \equiv \nabla^2 u$ on en cylinder with radius b. The top and bottom surfaces are isolated, whereas the circular surface $r = b$ (in cylindrical coordinates) obeys the cooling law $u + 2u'_r = 0$. Initial temperature is $u(r,0) = r^2 = x^2 + y^2$.

34. a) Find a bounded solution of the form $u(r,t) = v(r)e^{i\omega t}$ to the equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\Big(r\dfrac{\partial u}{\partial r}\Big) - \dfrac{n^2}{r^2}u, \quad 0 < r < a, \\ u(a,t) = e^{i\omega t}, \end{cases}$$

where $n \geq 0$ are integers. For which values $\omega > 0$ is there such a solution?

b) Let ω be such that the solution to a) exists. Study how this can be used to solve

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\Big(r\dfrac{\partial u}{\partial r}\Big) - \dfrac{n^2}{r^2}u, & 0 < r < a,\, t > 0, \\ u(a,t) = \sin \omega t, & u \text{ bounded}, \\ u(r,0) = 0, & u_t(r,0) = 0. \end{cases}$$

(Some eventual integrals need not to be calculated.)

35. Solve the Laplace equation $\nabla^2 u = 0$ in the cylinder $f = \sqrt{x^2+y^2} < R,\ 0 < z < L$, with $u = 0$ for $z = 0$ and $z = L$, and $u = \sin\frac{\pi z}{L}(1 - \cos\frac{\pi z}{L})$ for $r = R$.

36. Find the polynomial $P(x)$ of at most degree two which minimizes the integral $\displaystyle\int_0^\infty [\sqrt{x} - P(x)]^2 e^{-x}\,dx.$

37. Find the polynomial $P(x)$ of at most degree two which minimizes $\int_{-\infty}^{\infty}[x^4-P(x)]^2 e^{-x^2/2}dx$.

38. Find the plynomial $P(x)$ of at most degree two which minimizes the integral $\int_0^{\infty}[e^{x/4}-P(x)]^2 x e^{-x}dx$ as small as possible.

39. Find the polynomial of the form $P(x) = x^3 + ax^2 + bx + c$ which minimizes the integral $\int_0^1 [P(x)]^2 dx$

40. Prove att $\int_0^1 xP_{2m}(x)dx = \frac{1}{3}\binom{3/2}{m+1}$.

41. Find, (for example, by using the generating function), $H_n'(0)$, where H_n are the Hermites polynomials.

42. Prove the following formel for (the generalized) Laguerre polynomials $L_n^{\alpha}(x)$:

$$\frac{d}{dx}L_{n+1}^{\alpha}(x) = -L_n^{\alpha+1}(x).$$

(Tips: Use the generating function.)

43. Solve the Laplace equation $\Delta u = 0$ on the domain $x^2+y^2+z^2 < R^2$ with boundary condition $u = z(x^2+y^2)$ då $x^2+y^2+z^2 = R^2$.

44. Solve the following Laplace equation $\Delta u = 0$ on the domain $0 < a < r < b$ (with spherical coordinates[1])

$$\begin{cases} u = 1+\cos\theta, & \text{if } r = a, \\ u = \cos 2\theta, & \text{if } r = b. \end{cases}$$

45. Find a bounded solution to

$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = (1-2x^2)e^{-x^2}, & -\infty < x < \infty. \end{cases}$$

46. Solve the problem

$$\begin{cases} u''_{xx} + u''_{yy} = x, & 0 < x < 1,\ -\infty < y < \infty, \\ u'_x(0,y) = 0, \\ u(1,y) = ye^{-|y|} & . \end{cases}$$

47. Let f be in $L^2(\mathbf{R})$. Find a solution to

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & -\infty < x < \infty,\ 0 < y < a, \\ u(x,0) = 0, & u(x,a) = f(x). \end{cases}$$

Prove att

$$\int_{-\infty}^{\infty}|u(x,y)|^2 dx \le \int_{-\infty}^{\infty}|f(x)|^2 dx.$$

[1] $x = r\cos\phi\sin\theta$, boundary condition $y = r\sin\phi\sin\theta$, $z = r\cos\theta$

48. Find a periodic solution to the equation $y'' - y' + y = f'(t)$, where

$$f(t) = \begin{cases} 0 & \text{for} \quad 0 < t \le 1, \\ t - 1 & \text{for} \quad 1 < t < 2, \end{cases}$$

and f is periodic with period 2. ($f'(t)$ is considered in the distribution sense.)

49. Let f be defined by

$$f(t) = \begin{cases} -1 & \text{for} \quad 0 < t < 1, \\ 1 & \text{for} \quad 1 < t < 3, \end{cases}$$

and that let $f(t)$ be 3-periodic. Find $f'(t)$ (distributional derivative) and expand $f'(t)$ in complex trigonometric Fourier series. Use the result to compute the Fourier series expansion of $f(t)$.

50. Find the following function (that is, evaluate the sum)

$$f(\theta) = \sum_{1}^{\infty} \frac{\sin(2n-1)\theta}{(2n-1)^2}.$$

51. Find the complex Fourier series for the periodic function $f(x)$ which is equal to $x(x^2 - \pi^2)$. What is the sum of the series at the points 2π and $3\pi/2$?

52. Solve the problem

$$\begin{cases} u_{xx} + 1 = u_{tt}, \quad 0 < x < 2, t > 0 \\ u(0,t) = 0, \ u(x,0) = x - x^2, \\ u(2,t) = -2, \ u_t(x,0) = 0 \end{cases}$$

53. Solve the following Dirichlet problem on the unit disk $\{(x,y); x^2 + y^2 < 1\}$

$$u_{xx} + u_{yy} = 0, \ r = \sqrt{x^2 + y^2} < 1$$

where the function f is the function $f(\theta) = \sin^2\theta + \cos\theta$ (in polar coordinates)

54. Let

$$Q_n(x) = \frac{d^n}{dx^n}\left(x^n(1-x)^n\right), \quad n = 0, 1, 2, \ldots.$$

Q_n is a polynomial of degree n.

(a) Find the leading term and the constant term in $Q_n(x)$.

(b) Find the norm $\|Q_n\|$ av $Q_n(x)$ in $L^2(0,1)$.

(c) Prove att $Q_n(x)$ and $Q_m(x)$ are orthogonal in $L^2(0,1)$ if $n \neq m$.

55. The function $f(x)$ is continuous and has Fourier transform $\widehat{f}(\xi) = \frac{\ln(1+\xi^2)}{\xi^2}$. Find $f(0)$ and $\int_{-\infty}^{\infty} f(x)dx$

56. Find the solution $f(t)$, $t > 0$, to the equation

$$f''(t) - f'(t) + f(t) + 3\int_0^t f(\tau)d\tau = 2e^t$$

with initial condition $f(0) = 1, f'(0) = 0$.

57. Let $u(x,t)$ be the solution to the following equation with initial condition

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad t>0, 0<x<\pi \\ u(0,t) = u(\pi,0) = 0, \\ u(x,0) = 0,\ u_t(x,0) = g(x) \end{cases}$$

Prove att for $t > 0$,

$$\int_0^\pi |u_t(x,t)|^2 dx \le \int_0^\pi |g(x)|^2 dx.$$

(Tip: Use variable separation).

58. For which k can you guarantee $f \in C^{(k)}$, where

$$(a) \qquad f(\theta) = \sum_0^\infty \frac{\cos(n\theta)}{3^n}$$

$$(b) \qquad f(\theta) = \sum_0^\infty \frac{\cos(2^n\theta)}{3^n}$$

Answers

1. $f(x) = \frac{4}{3} + \frac{2}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n(1+in\pi)}{n^2} e^{in\pi x}, \quad y = -\frac{4}{3} + \frac{2}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1}(1+in\pi)}{n^2(2n^2\pi^2 + in\pi + 1)} e^{in\pi x}$

2. $y(t) = \frac{2}{9} - \sum_{n=1}^{\infty} \frac{3(1-\cos\frac{2n\pi}{3})}{\pi^2 n^2(3 - \frac{4}{9}n^2\pi^2)} \cos\frac{2n\pi t}{3}$

3. $\cos x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nx \quad (0 < x < \frac{\pi}{2})$. The sum is $\frac{\pi^2}{64}$.

4. $u(x,t) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{n^2\pi^2} e^{-\sqrt{\frac{n\pi}{2}}x} \cos(n\pi t - \sqrt{\frac{n\pi}{2}}xx)$

5. $\frac{2}{2\ln 2} \ln r + \frac{2}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n_1}}{n^2(2^n - 2^{-n})} (r^n - r^{-n}) e^{in\theta}$

6. a) $-\frac{i\pi}{2a}\omega e^{-a|\omega|}$ b) $\frac{\pi}{2a^3}(1+a|\omega|)e^{-a|\omega|}$
   c) $\frac{\pi}{10} e^{-|\omega|}(1 - 2isgn\omega) - \frac{\pi}{10} e^{-2|\omega|} e^{i\omega} \left(\frac{3}{2} - 2isgn\omega\right)$
   d) $-\frac{4iab\omega}{(\omega^2 + 2b\omega + a^2 + b^2)(\omega^2 - 2b\omega + a^2 + b^2)}$

7. a) i b) $\dfrac{i}{2\sqrt{2}}$

8. $\dfrac{1}{2}$

9. $\pi(1-e^{-1})$

10. $\dfrac{1}{9\pi}$

11. $\hat{f}(\omega) = \dfrac{2\pi\sqrt{\omega}}{1+\omega}$ for $0 < \omega < 2$, 0 otherwise. a) $\dfrac{\pi}{2}$, b) $2\pi\Big(\ln 3 - \dfrac{2}{3}\Big)$

12. $\dfrac{\pi}{8}(e^2+1)$

13. $\theta(t)e^{-2t}\cos t$

14. $u(t) = \dfrac{1}{2}e^{-t/\sqrt{3}}\theta(t) + \dfrac{1}{2}e^{\sqrt{3}t}(1-\theta(t))$

15. $\dfrac{3}{4}e^{-2|t|} - te^{-2t}\theta(t)$

16. $h(t) = \dfrac{1}{2\pi}\dfrac{t}{(1+t^2)^2}$; $x(t) = \cos\omega t$ ger $y(t) = \dfrac{\omega}{4}e^{-|\omega|}\sin\omega t$; not causal, stable.

17. $\dfrac{\pi}{2\sqrt{6}}e^{-5/96}$

18. $\sqrt{\dfrac{2}{\pi}}\displaystyle\sum_{n=-\infty}^{\infty}\Big[1-\dfrac{1}{2}(-1)^n\Big]e^{-n^2\pi^2/8+2|n|\pi}e^{in\pi t}$

19. The sum is $k(N-k)$.

20. $X(\mu) = \displaystyle\sum_{n=0}^{N-1} x(n)e^{-2\pi i\mu n/N} = \dfrac{\sin\frac{\pi}{N}}{\cos\frac{2\mu\pi}{N} - \cos\frac{\pi}{N}}$

21. $c_n = \begin{cases} \pi(e^{\frac{1}{2}} - e^{-\frac{1}{2}})e^{-|n|} & \text{for } n \neq 0 \\ 2\pi(1-e^{-\frac{1}{2}}) & \text{for } n = 0 \end{cases}$

22. $\dfrac{2\sinh 1}{\frac{1}{2}\sinh 2 + 1}\cosh x + \dfrac{2e^{-1}}{\frac{1}{2}\sinh 2 - 1}\sinh x$

23. Egenvalue: $\lambda_k = \nu_k^2$, where ν_k are the positive roots to the equation $\tan\nu a = \frac{3\nu}{2\nu^2-1}$

24. $\lambda_1 = r - \beta_1^2$, where β_1 is the positive root of equ. $\tanh\beta = \dfrac{\beta}{2}$; $u_1(x) = e^{-2x}\sinh\beta_1 x$
$\lambda_n = 4 + \beta_n^2$, where β_n, $n = 2, 3, \ldots$, are the positive roots of equ. $\tan\beta = \dfrac{\beta}{2}$; $u_n(x) = e^{-2x}\sin\beta_n x$
$e^{-2x} = \displaystyle\sum_{n=1}^{\infty}\dfrac{2\sqrt{\lambda_N}[\sqrt{\lambda_n} + 2(-1)^n]}{\beta_n(\lambda_n - 2)}u_n(x)$

25. $u(x,y) = \dfrac{1}{6}(y^3-y) + \dfrac{2}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^3\sinh 2n\pi}(\sinh n\pi x + 7\sinh n\pi(2-x))\sin n\pi y$

26. $u(x,t) = 1 - x + \dfrac{8}{\pi^3}\sum_{k=0}^{\infty}\dfrac{1}{(2k+1)^3}e^{-\frac{2}{3}(2k+1)^2\pi^2[(1+t)^{3/2}-1]}\sin(2k+1)\pi x$

27. $u(x,y) = -\dfrac{8}{\pi^3}\sin\pi x\dfrac{\sin(\sqrt{20-\pi^2}y)}{\sin\sqrt{20-\pi^2}} -$
$- \dfrac{8}{\pi^3}\sum_{k=1}^{\infty}\dfrac{1}{(2k+1)^3}\sin(2k+1)\pi x\dfrac{\sinh(\sqrt{(2k+1)^2\pi^2-20}y)}{\sinh\sqrt{(2k+1)^2\pi^2-20}}$

28. $u(x,t) = e^{-4\pi^2 t}\sin 2\pi x + 2\pi\sin 1\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2\pi^2-1}\left[\dfrac{t}{n^2\pi^2} - \dfrac{1}{n^4\pi^4}(1-e^{-n^2\pi^2 t})\right]\sin n\pi x$

29.

$$\begin{aligned} u(x,t) &= (t+1)(1-x) + \sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}(e^{-2n^2\pi^2 t}-1)\sin n\pi x = \\ &= (t+1)(1-x) + \frac{x^2}{4} - \frac{x}{6} - \frac{x^3}{12} + \sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}e^{-2n^2\pi^2 t}\sin n\pi x \end{aligned}$$

30. $u(r,\theta) = \sum_{n=0}^{\infty}\dfrac{2[(n+\frac{1}{2})\frac{\pi}{\ln 2}(-1)^n - 1]}{(n+\frac{1}{2})\pi[(n+\frac{1}{2})^2(\frac{\pi}{\ln 2})^2+1]}\,\dfrac{\sinh(n+\frac{1}{2})\frac{\pi\theta}{\ln 2}}{\sinh(n+\frac{1}{2})\frac{\pi^2}{4\ln 2}}\sin(n+\dfrac{1}{2})\dfrac{\pi\ln r}{\ln 2}$

31. $\sin(2\sin x) = 2\sum_{k=0}^{\infty}J_{2k+1}(2)\sin(2k+1)x$

32. $\dfrac{qc^2}{S\omega^2}\left(\dfrac{J_0\left(\frac{\omega r}{c}\right)}{J_0\left(\frac{\omega a}{c}\right)} - 1\right)\sin\omega t$

res. frequences is $\dfrac{c}{a}\alpha_{0,n}$, where $\alpha_{0,n},\ n = 1, 2, \ldots$, is J_0:s positive root.

33. $u(r,t) = \sum_{k=1}^{\infty}\dfrac{8b^2[(2+\frac{b}{2})\alpha_k^2 - 2b]}{\alpha_k^2(4\alpha_k^2+b^2)J_0(\alpha_k)}\,e^{-(\frac{\alpha_k}{b})^2 t}J_0\left(\dfrac{\alpha_k r}{b}\right)$, where α_k are the pos. roots of $J_0(\alpha) + \dfrac{2}{b}\alpha J_0'(\alpha) = 0$.

34. a. $v(r) = \dfrac{J_n(\omega r)}{J_n(\omega a)}$ if $J_n(\omega a) \neq 0$.

b. $u(r,t) = \dfrac{J_n(\omega r)}{J_n(\omega a)}\sin\omega t + \sum_{k=1}^{\infty}a_k\sin\dfrac{\alpha_k t}{a}J_N\left(\dfrac{\alpha_k r}{a}\right)$, where α_k are the positive roots of $J_n(x)$, and

$$a_k = -\frac{2\omega}{a\alpha_k[J_{n+1}(\alpha_k)]^2 J_n(\omega a)}\int_0^a J_n(\omega r)J_n\left(\frac{\alpha_k r}{a}\right)r\,dr\left(= \frac{2\omega a}{(\omega^2 a^2 - \alpha_k^2)J_{n+1}(\alpha_k)}\right)$$

35. $u(r,z) = \dfrac{I_0(\frac{\pi r}{L})}{I_0(\frac{\pi R}{L})}\sin\dfrac{\pi z}{L} - \dfrac{1}{2}\dfrac{I_0(\frac{2\pi r}{L})}{I_0(\frac{2\pi R}{L})}\sin\dfrac{2\pi z}{L}$

36. $\dfrac{\sqrt{\pi}}{16}(3 + 6x - \dfrac{1}{2}x^2)$

37. $3(2x^2 + 12)$

38. $\dfrac{8}{81}(x^2 + 12)$

39. $x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$

41. $H'_{2k}(0) = 0, \quad H'_{2k+1}(0) = 2(-1)^k \frac{2k+1)!}{k!}$, $k = 0, 1, \ldots$

43. $u = \frac{2}{5}R^2 z + \frac{3}{5}(x^2 + y^2 + z^2)z - z^3$

44. $u(r,\theta) = \frac{1}{3(b-a)}\Big(\frac{4ab}{r} - 3a - b\Big) + \frac{a^2}{b^3 - a^3}\Big(\frac{b^3}{r^2} - r\Big)\cos\theta +$
$+ \frac{2b^3}{3(b^5 - a^5)}\Big(r^2 - \frac{a^5}{r^3}\Big)(3\cos^2\theta - 1)$

45. $u(x,t) = \frac{4kt + 1 - 2x^2}{(4kt+1)^{5/2}} e^{-\frac{x^2}{4kt+1}}$

46. $u(x,y) = \frac{1}{6}(x^3 - 1) + \frac{4}{\pi}\int_0^\infty \frac{\eta}{(1+\eta^2)^2}\frac{\cosh \eta x}{\cosh \eta} \sin \eta y d\eta$

47. $u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh \xi y}{\sinh \xi a}\hat{f}(\xi) e^{i\xi x} d\xi \Bigg(= \frac{1}{2a}\int_{-\infty}^{\infty} \frac{\sin \frac{\pi y}{a}}{\cosh \frac{\pi(x-t)}{a} + \cos \frac{\pi y}{a}} f(t)dt \Bigg)$

48. $y(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{n\pi - i(1 - (-1)^n)}{2n\pi(n^2\pi^2 - 1 + in\pi)} e^{in\pi t}$

49. $f'(t) = 2\sum_{n=-\infty}^{\infty} [\delta(t - 3n - 1) - \delta(t - 3n)] = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{3}(e^{-in\frac{2\pi}{3}} - 1)e^{in\frac{2\pi}{3}t}$

$f(t) = \frac{1}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{-in\frac{2\pi}{3}} - 1}{in\pi} e^{in\frac{2\pi}{3}t}$

50. $f(\theta) = -\frac{\pi}{4}(\frac{1}{2}\theta^2 - \frac{\pi}{2}\theta)$, if $\theta > 0$ and $f(\theta) = -f(-\theta)$ if $\theta > 0$.

51. $\frac{1}{12}\sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n^3}$. 0 resp $\frac{3}{8}\pi^3$.

52. $u(x,t) = -\frac{x^2}{2} + \frac{16}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\cos((2k+1)\pi t)\sin(\frac{(2k+1)\pi}{2}x)$

53. $u(x,y) = -\frac{1}{4}(x^2 - y^2 - 2) + \frac{1}{2}x$ eller $u(r,\theta) = -\frac{1}{4}(r^2\cos 2\theta - 2) + \frac{1}{2}r\cos\theta$ (in polar coordinates).

54. (a) Leading term is $(-1)^n(2n)(2n-1)\ldots(n+1)$, constanta term is $n!$. (b) $\sqrt{(2n)!}$.

55. $f(0) = 1$, $\int_{-\infty}^{\infty} f(x)dx = 1$.

56. $f(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$.