MATHEMATICAL SCIENCES
Chalmers and Göteborg University

Date: 22 October 2005
Material allowed: only the three attached pages with formulas
Time: 8.30 – 13.30
Christoffer Cromvik,
ph. 0762-721860, will come at about 9.30 and 12.30

Exam in Fourier Analysis

MAN 530, TMA362

1. State and prove the theorem about the best approximation of a function $f \in \mathrm{PC}[a,b]$ by means of the functions of an orthonormal system $\{\phi_1, ...\phi_N\}$.

2. Deduce by means of the Fourier transform the solution of the heat equation in the half-plane $\{(x,t) : t > 0\}$, with given initial values $f(x)$ for $t = 0$..

3. Find the Fourier series of the function $f(x) = x^3$ in the interval $[-\ell, \ell]$, for example in the following way: Choose a constant c in such a way that the Fourier series of $g(x) = f(x) - cx$ kan be differentiated term by term, and determine first the Fourier series of g'.

4. Solve the initial value problem

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \qquad 0 < x < \pi, \quad t > 0 \\ u(0,t) &= 0, \quad u(\pi,t) = 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= x \sin x. \end{aligned}$$

   Here $c > 0$ is a constant.

5. Solve the following Dirichlet problem in the rectangle $0 < x < \ell$, $0 < y < L$

$$\begin{aligned} \Delta u &= 0 \\ u(0,y) &= 0, \quad u(\ell,y) = 1 \\ u(x,0) &= 1, \quad u(x,L) = 1. \end{aligned}$$

6. How many eigenvalues λ with $\lambda < 50$ does the following Sturm-Liouville problem have?

$$f'' + \lambda f = 0, \qquad f(0) = 0, \qquad f'(1) = -f(1)$$

The grading will be finished by 8 November. You may then get your exam paper at the reception (mottagningsrummet) weekdays from 12.30 to 13.00.

Solutions to
exam in Fourier Analysis

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3.

The function $g(x) = x^3 - cx$ is defined and odd in the interval $[-\ell, \ell]$. In order to allow differentiation term by term of its Fourier series, we must choose the value of c so as to make the 2ℓ-periodic extension of g continuous on the line. This will be true if $g(\ell) = 0$, which implies $c = \ell^2$. With c determined in this way, g will extend to a piecewise smooth and continuous function on the line, and the theorem about differentiation of Fourier series applies. So if the Fourier series of g is

$$g(x) = \sum_1^\infty b_n \sin \frac{n\pi x}{\ell},$$

that of g' is

$$g'(x) = \sum_1^\infty \frac{n\pi}{\ell} b_n \cos \frac{n\pi x}{\ell}.$$

But $g'(x) = 3x^2 - \ell^2$, and the Fourier series of this even function can be obtained from entry 16 of the Fourier series table, with $\theta = \pi x/\ell$. One finds

$$3x^2 - \ell^2 = \frac{12\ell^2}{\pi^2} \sum_1^\infty \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{\ell}.$$

Identifying coefficients, we have

$$b_n = \frac{12\ell^3}{\pi^3} \frac{(-1)^n}{n^3}.$$

Thus we have the Fourier series of g, and we need also that of the term $\ell^2 x$. Entry 1 of the table shows that

$$\ell^2 x = \frac{2\ell^3}{\pi} \sum_1^\infty \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{\ell}.$$

Summing up, we get

$$x^3 = \ell^3 \sum_1^\infty (-1)^n \left(\frac{12}{\pi^3 n^3} - \frac{2}{\pi n} \right) \sin \frac{n\pi x}{\ell}.$$

It is of course also possible to compute the coefficients by straightforward integration.

4.

This is a standard problem for the wave equation, and one knows that the solution is given by an expression of type

$$u(x,t) = \sum_1^\infty b_n \sin nx \, \sin nct.$$

Notice that there will be no cosine term in nct, since the initial values $u(x,0)$ are 0. The initial values of u_t lead to

$$\sum_1^\infty ncb_n \sin nx = x \sin x, \quad 0 < x < \pi.$$

To find the Fourier sine series expansion of $x \sin x$, one can compute the coefficients directly. But we choose another method, and start by expanding x in a cosine series. From entry 2 in the table, we have

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^\infty \frac{\cos(2n-1)x}{(2n-1)^2}, \qquad 0 < x < \pi.$$

Now $\cos(2n-1)x \, \sin x = (\sin 2nx - \sin(2n-2)x)/2$, and so

$$x \sin x = \frac{\pi}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^\infty \frac{\sin 2nx}{(2n-1)^2} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{\sin(2n-2)x}{(2n-1)^2}.$$

In the last sum here, we replace n by $n+1$. Since the first term is 0, the sum equals

$$\sum_{n=1}^\infty \frac{\sin 2nx}{(2n+1)^2}.$$

Inserting this above, we get

$$\begin{aligned} x \sin x &= \frac{\pi}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^\infty \left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right) \sin 2nx \\ &= \frac{\pi}{2} \sin x - \frac{16}{\pi} \sum_{n=1}^\infty \frac{n}{(4n^2-1)^2} \sin 2nx. \end{aligned}$$

We conclude that $b_1 = \pi/2c$, that $b_n = 0$ for odd $n > 1$ and that

$$b_{2n} = -\frac{8}{\pi c(4n^2-1)^2}.$$

Finally, we get

$$u(x,t) = \frac{\pi}{2c}\sin x \sin ct - \frac{8}{\pi c}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\sin 2nx\ \sin 2nct.$$

5.

Consider first the problem with the boundary condition $u(\ell, y) = 1$ replaced by the homogeneous condition $u(\ell, y) = 0$. By separation of variables, one finds that the solution u_1 of this problem is of the form

$$u_1(x,y) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}\left(a_n \sinh\frac{n\pi y}{\ell} + b_n \sinh\frac{n\pi(L-y)}{\ell}\right).$$

The coefficients a_n and b_n are determined by the boundary conditions for $y = 0$ and $y = L$. This leads to

$$\sum_{n=1}^{\infty} b_n \sinh\frac{n\pi L}{\ell}\sin\frac{n\pi x}{\ell} = 1$$

and the same equation for a_n. The expansion in a sine series of the constant 1 is (entry 6 in the table)

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{\ell}, \qquad 0 < x < \ell.$$

Thus

$$a_{2n-1} = b_{2n-1} = \frac{4}{\pi}\,\frac{1}{(2n-1)\sinh(2n-1)\pi L/\ell}, \qquad n = 1, 2, ...$$

and

$$a_{2n} = b_{2n} = 0, \qquad n = 1, 2, ...,$$

so that

$$u_1(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh(2n-1)\pi L/\ell}\cdot$$

$$\cdot\sin\frac{(2n-1)\pi x}{\ell}\left(\sinh\frac{(2n-1)\pi y}{\ell} + \sinh\frac{(2n-1)\pi(L-y)}{\ell}\right).$$

Next we consider the Dirichlet problem obtained by replacing in the given problem the boundary conditions $u(x,0) = u(x,L) = 1$ by $u(x,0) = u(x,L) = 0$. To find the solution u_2, we can apply the preceding method after swapping the variables x and y and also the lengths ℓ and L. Thus we have

$$u_2(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{L}\left(a'_n \sinh\frac{n\pi x}{\ell} + b'_n \sinh\frac{n\pi(\ell - x)}{L}\right).$$

Because of the conditions for $x = 0$ and $x = \ell$, we get expressions for a'_n analogous to those of a_n above, whereas all $b'_n = 0$. This means that

$$u_2(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh(2n-1)\pi\ell/L} \sin\frac{(2n-1)\pi y}{L} \sinh\frac{(2n-1)\pi x}{L}.$$

Since the solution u of the given problem is $u = u_1 + u_2$, we finally find

$$\begin{aligned} u(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty} &\frac{1}{(2n-1)\sinh(2n-1)\pi L/\ell} \sin\frac{(2n-1)\pi x}{\ell} \cdot \\ &\cdot\left(\sinh\frac{(2n-1)\pi y}{\ell} + \sinh\frac{(2n-1)\pi(L-y)}{\ell}\right) \\ +\frac{4}{\pi}\sum_{n=1}^{\infty} &\frac{1}{(2n-1)\sinh(2n-1)\pi\ell/L} \sin\frac{(2n-1)\pi y}{L} \sinh\frac{(2n-1)\pi x}{L}. \end{aligned}$$

We remark that a somewhat simpler method is to observe that the boundary values 1 on all four sides of the rectangle would trivially give a solution which is identically 1 in the rectangle. Thus $1 - u$ solves a Dirchlet problem with value 1 on one rectangle side and 0 on the other three sides. In this way, one can write expressions for $1 - u$ and u of the above type but shorter.

6.

In the case $\lambda < 0$ we let $\lambda = -\mu^2$ with $\mu > 0$. Then the general solution of the differential equation is $f(x) = a\cosh\mu x + b\sinh\mu x$. The first boundary condition implies $a = 0$, and the second leads to $b\mu\cosh\mu = -b\sinh\mu$. We get $\tanh\mu = -\mu$, an equation which has no solution $\mu > 0$. Thus there are no negative eigenvalues.

For $\lambda = 0$ we get solutions $f(x) = ax + b$, and the boundary conditions imply $b = 0$ and $a = -a$, so that $a = b = 0$. This means that 0 is not an eigenvalue.

For $0 < \lambda < 50$ we let $\lambda = \nu^2$ with $0 < \nu < \sqrt{50}$. The solutions of the differential equation are $f(x) = a\cos\nu x + b\sin\nu x$, and the boundary conditions imply $a = 0$ and $b\nu \cos\nu = -\sin\nu$, so that $\tan\nu = -\nu$. We must thus determine the number of solution of the equation $\tan\nu = -\nu$ with $0 < \nu < \sqrt{50}$. In the ν, y plane, this means the number of intersections between the line $y = -\nu$ and the curve $y = \tan\nu$, with ν in the indicated interval. It is immediately seen that the line has precisely one intersection with each branch of the tan curve, and in particular one intersection in each interval $(n - 1/2)\pi < \nu < n\pi, \quad n = 1, 2, \ldots.$ We must thus determine how many of these intersections that fall in $0 < \nu < \sqrt{50}$. But $\sqrt{50} > 2\pi$, since (equivalently) $\pi^2 < 12.5$. And $\sqrt{50} < 2.5\pi$, since $\pi^2 > 8$. This means that we get the first two intersections, and the final answer is two.