MATHEMATICAL SCIENCES
Chalmers and Göteborg University

Date: 9 January 2006
Material allowed: only the three attached pages with formulas
Time: 8.30 – 13.30
Henrik Seppänen,
ph. 0762-721860, will come at about 9.30 and 12.30

# Exam in Fourier Analysis
MAN 530, TMA362

1. Let the function $f$ be piecewise continuous in the interval $[-\pi, \pi]$. Deduce the formula for the complex Fourier coefficients of $f$ and the relations between the real and complex coefficients. Also show how one by means of Fourier series in $[-\pi, \pi]$ can derive the expansion in a Fourier cosine series of a function in the interval $[0, \pi]$, and do the same for the sine series.

2. Assume that $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal system in $\mathrm{PC}[a, b]$. Define completeness of this system and state the theorem saying that completeness is equivalent to each of two other properties of $(\phi_n)_{n\in\mathbb{N}}$.

3. Compute the Fourier series of the function $e^{iax}$ in the interval $[-1, 1]$, in real form, i.e., with cosine and sine functions. Here $a > 0$.

4. Compute the convolution
$$e^{-x^2/3} * e^{-x^2/4} * e^{-x^2/5},$$
for instance by means of Fourier transforms.

5. Solve the initial value problem
$$\begin{aligned} u_t(x,t) &= k\,u_{xx}(x,t) + \sin x, \qquad 0 < x < \pi, \quad t > 0 \\ u(0,t) &= 0, \quad u(\pi,t) = 0 \\ u(x,0) &= 2 - \sin x. \end{aligned}$$
Here $k > 0$ is a constant.

Please turn over

6. Let $r$ and $\theta$, $-\pi < \theta \leq \pi$, denote polar coordinates in the plane. Solve the following Dirichlet problem in the ring $\{R_0 < r < R_1\}$.

$$\begin{aligned} \Delta u &= 0, \qquad R_0 < r < R_1 \\ u(R_0, \theta) &= 0 \text{ for all } \theta, \\ u(R_1, \theta) &= 1 \text{ if } |\theta| < \pi/2 \text{ and } 0 \text{ otherwise.} \end{aligned}$$

In polar coordinates

$$\Delta u = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta}.$$

The grading will be finished by 26 January. You may then see your exam paper in the office of the new building of Mathematical Sciences weekdays from 8.30 to 13.00.

**Solutions to**
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3.

One can compute the Fourier sine and cosine coefficients directly, but we shall here go via the complex Fourier coefficients, to get slightly simpler integrals. In the interval $[-1, 1]$, the functions to be used are then $e^{in\pi x}$. We observe first of all that if $a$ is an integer multiple of $\pi$, say $a = N\pi$, then

$$e^{iax} = \cos N\pi x + i \sin N\pi x.$$

This is already the desired Fourier series in this case.

So assume now that $a$ is not a multiple of $\pi$. Then the complex Fourier coefficients of the given function are

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} e^{iax} e^{-in\pi x}\, dx = \frac{1}{2i(a-n\pi)} \left(e^{i(a-n\pi)} - e^{-i(a-n\pi)}\right) \\ &= \frac{1}{a-n\pi}\sin(a-n\pi) = (-1)^n \frac{\sin a}{a-n\pi}, \end{aligned}$$

for $n \in \mathbb{Z}$. Using the relations between the real and complex Fourier coefficients, we get for $n > 0$

$$a_n = c_n + c_{-n} = (-1)^n \sin a \left(\frac{1}{a-n\pi} + \frac{1}{a+n\pi}\right) = (-1)^n \sin a\, \frac{2a}{a^2 - n^2\pi^2}$$

and

$$b_n = i(c_n - c_{-n}) = i(-1)^n \sin a \left(\frac{1}{a-n\pi} - \frac{1}{a+n\pi}\right) = i(-1)^n \sin a\, \frac{2n\pi}{a^2 - n^2\pi^2}.$$

Moreover,

$$a_0 = 2c_0 = 2\,\frac{\sin a}{a}.$$

This means that the Fourier series of $e^{iax}$ is

$$\frac{\sin a}{a} + 2(-1)^n \sin a \sum_{1}^{\infty} \frac{1}{a^2 - n^2\pi^2}(a\cos n\pi x + in\pi \sin n\pi x).$$

4.

The Fourier transform of this convolution is the product of the Fourier transforms of the three factor functions. From the table, we know that the transform of $e^{-ax^2/2}$ is $\sqrt{2\pi/a}\, e^{-\xi^2/2a}$. So the product of the three transforms is

$$\sqrt{3\pi}\, e^{-3\xi^2/4}\sqrt{4\pi}\, e^{-4\xi^2/4}\sqrt{5\pi}\, e^{-5\xi^2/4} = 2\pi^{3/2}\sqrt{15}\, e^{-12\xi^2/4}.$$

But this is similarly the Fourier transform of the function

$$\pi\sqrt{5}e^{-x^2/12}.$$

The given convolution coincides with this function, since the two have the same Fourier transform.

5.

The equation is inhomogeneous because of the term $\sin x$. Since this term does not depend on $t$, one can use a steady-state solution $u_0(x)$. Then $u_0$ should satisfy the equation, so that $0 = k(u_0)_{xx} + \sin x$. Integrating twice, we get $u_0(x) = k^{-1}\sin x + ax + b$. Here one determines the constants $a$ and $b$ so as to make $u_0$ fulfill the boundary conditions $u_0(0) = u_0(\pi) = 0$. This leads to $b = 0$ and $a = 0$, and thus $u_0(x) = k^{-1}\sin x$. The difference $v(x,t) = u(x,t) - u_0(x)$ will now satisfy

$$\begin{aligned} v_t(x,t) &= k\, v_{xx}(x,t) \\ v(0,t) &= 0, \quad v(\pi,t) = 0 \\ v(x,0) &= 2 - (1 + k^{-1})\sin x. \end{aligned}$$

This is a standard problem. We expand the initial value function $2 - (1 + k^{-1})\sin x$ in a sine series in $[0,\pi]$. Using entry 6 in the table to expand the constant function and observing that the second term $-(1+k^{-1})\sin x$ already has the right form, we get

$$2 - (1 + k^{-1})\sin x = \left(\frac{8}{\pi} - 1 - \frac{1}{k}\right)\sin x + \frac{8}{\pi}\sum_{n=2}^{\infty}\frac{\sin(2n-1)x}{2n-1}.$$

Then the solution $v$ is given by

$$v(x,t) = \left(\frac{8}{\pi} - 1 - \frac{1}{k}\right)e^{-kt}\sin x + \frac{8}{\pi}\sum_{n=2}^{\infty}\frac{1}{2n-1}e^{-k(2n-1)^2 t}\sin(2n-1)x.$$

This means that the solution $u = v + u_0$ of the given problem is

$$u(x,t) = \left(\left(\frac{8}{\pi} - 1 - \frac{1}{k}\right)e^{-kt} + \frac{1}{k}\right)\sin x + \frac{8}{\pi}\sum_{n=2}^{\infty}\frac{1}{2n-1}e^{-k(2n-1)^2 t}\sin(2n-1)x.$$

6.

We know that separation of variables produces an expression for the solution $u$ of the form

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_{n\neq 0} e^{in\theta}(a_n r^n + b_n r^{-n}).$$

The coefficients can be determined by means of the boundary conditions. Letting $r = R_0$, we get

$$a_0 + b_0 \ln R_0 + \sum_{n\neq 0} e^{in\theta}(a_n R_0^n + b_n R_0^{-n}) = 0$$

for all $\theta$. But this is a Fourier series in $\theta$, so all its coefficients must be 0. Thus

$$\begin{aligned} a_0 + b_0 \ln R_0 &= 0 && (1)\\ a_n R_0^n + b_n R_0^{-n} &= 0, \qquad n \neq 0. && (2) \end{aligned}$$

For $r = R_1$, we need to expand the given boundary value function $\chi_{\{|\theta|<\pi/2\}}$ in a Fourier series. Entry 12 in the table, with $a = \pi/2$ and after multiplication by $2a = \pi$, tells us that

$$\chi_{\{|\theta|<\pi/2\}} = \frac{1}{2} + \frac{2}{\pi}\sum_{1}^{\infty}\frac{\sin n\pi/2}{n}\cos n\theta.$$

Now $\sin n\pi/2$ is 0 for even $n$ and equals $(-1)^{k+1}$ for $n = 2k-1$. Rewriting the cosine function in term of exponentials with imaginary exponents, we get

$$\chi_{\{|\theta|<\pi/2\}} = \frac{1}{2} + \frac{1}{\pi}\sum_{1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\left(e^{i(2k-1)\theta} + e^{-i(2k-1)\theta}\right).$$

This Fourier series must coincide with the one obtained by setting $r = R_1$ in the series for $u$ above. Therefore,

$$a_0 + b_0 \ln R_1 = \frac{1}{2} \qquad (3)$$

and

$$a_{2k-1}R_1^{2k-1} + b_{2k-1}R_1^{-(2k-1)} = \frac{(-1)^{k+1}}{\pi(2k-1)}, \qquad k = 1, 2, ... \tag{4}$$

and

$$a_{-(2k-1)}R_1^{-(2k-1)} + b_{-(2k-1)}R_1^{2k-1} = \frac{(-1)^{k+1}}{\pi(2k-1)}, \qquad k = 1, 2, .... \tag{5}$$

For even $n \neq 0$, we see that $a_n R_1^n + b_n R_1^{-n} = 0$. Together with (2), this implies that $a_n = b_n = 0$ for such $n$. To get the other coefficients, we first observe that (1) and (3) easily lead to

$$a_0 = -\frac{\ln R_0}{2 \ln R_1/R_0} \qquad \text{and} \qquad b_0 = \frac{1}{2 \ln R_1/R_0}.$$

Combining (2) and (4), one finds that

$$a_{2k-1} = \frac{(-1)^{k+1}}{\pi(2k-1)} \frac{R_1^{2k-1}}{R_1^{2(2k-1)} - R_0^{2(2k-1)}}, \qquad k = 1, 2, ...$$

and

$$b_{2k-1} = \frac{(-1)^{k}}{\pi(2k-1)} \frac{R_0^{2(2k-1)} R_1^{2k-1}}{R_1^{2(2k-1)} - R_0^{2(2k-1)}}, \qquad k = 1, 2, ....$$

To simplify the computation of $a_{-(2k-1)}$ and $b_{-(2k-1)}$, we observe that (2) and (5) say that the couple $(a_{-(2k-1)}, b_{-(2k-1)})$ satisfies the same two equations as the couple $(b_{2k-1}, a_{2k-1})$, for $k = 1, 2, \ldots$. Thus $a_{-(2k-1)} = b_{2k-1}$ and $b_{-(2k-1)} = a_{2k-1}$. Notice that this implies

$$a_{-(2k-1)}r^{-(2k-1)} + b_{-(2k-1)}r^{2k-1} = a_{2k-1}r^{2k-1} + b_{2k-1}r^{-(2k-1)}.$$

Inserting the above expressions for $a_{2k-1}$ and $b_{2k-1}$, we find that

$$a_{2k-1}r^{2k-1} + b_{2k-1}r^{-(2k-1)} = \frac{(-1)^{k+1}}{\pi(2k-1)} \frac{R_1^{2k-1}r^{2k-1} - R_0^{2(2k-1)}R_1^{2k-1}r^{-(2k-1)}}{R_1^{2(2k-1)} - R_0^{2(2k-1)}}$$

In the formula for $u$, we can combine the terms with $n$ and $-n$ and get the final result

$$\begin{aligned} u(r,\theta) = & \quad \frac{\ln r/R_0}{2 \ln R_1/R_0} \\ & + \quad \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \frac{R_1^{2k-1}r^{2k-1} - R_0^{2(2k-1)}R_1^{2k-1}r^{-(2k-1)}}{R_1^{2(2k-1)} - R_0^{2(2k-1)}} \cos(2k-1)\theta. \end{aligned}$$