Lecture notes for Topology MMA100

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1 The fundamental group

1.1 Paths

A path α from the point x to the point y in a space X is a map $\alpha : I \to X$, such that $\alpha(0) = x$ and $\alpha(1) = y$. We find it convenient to write this symbolically as $x \overset{\alpha}{\to} y$, though α is of course not a map from the point x to the point y.

If $x \overset{\alpha}{\to} y$, then α^{-1} is the path $y \to x$ defined by $\alpha^{-1}(t) = \alpha(1-t)$.

The constant path c_x at the point x in X is defined by $c_x(t) = x$ for all $t \in I$. It is a path $x \to x$.

Paths $x \overset{\alpha}{\to} y \overset{\beta}{\to} z$ can be composed to a path $x \overset{\alpha.\beta}{\to} z$, by defining

$$\alpha.\beta(t) = \begin{cases} \alpha(2t) & \text{if} \quad 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

The continuity of $\alpha.\beta$ follows from the Gluing lemma in Armstrong's book (or can easily be checked by hand).

This composition is not associative in general.

If $f : X \to Y$ is a map and $x \overset{\alpha}{\to} y$ is a path in X, then $f \circ \alpha$ is a path $f(x) \to f(y)$ in Y.

We note that it is immediate from the definition of the composition that $f \circ (\alpha.\beta) = (f \circ \alpha).(f \circ \beta)$.

1.2 Homotopy classes of paths

Two paths $x \overset{\alpha,\beta}{\to} y$ in X are (path-)homotopic, $\alpha \simeq \beta$, if there is a map $F : I \times I \to X$, such that, for all t and s,

$$F(t,0) = \alpha(t),\ F(t,1) = \beta(t),\ F(0,s) = x, \text{ and } F(1,s) = y.$$

The map F is then called a (path-)homotopy from α to β, written $F : \alpha \simeq \beta$. Note that if s is fixed the map $F_s(t) = F(t,s)$ is a path $x \to y$ and, with this notation, $\alpha = F_0$, while $\beta = F_1$.

Theorem 1.1 *The relation $\simeq$ is an equivalence relation on the set of paths $x \to y$.*

Proof. Suppose $F : \alpha \simeq \beta$. Define $F^{-1}(t,s) = F(t, 1-s)$. Then $F^{-1} : \beta \simeq \alpha$. This shows symmetry.

Suppose $F : \alpha \simeq \beta$ and $G : \beta \simeq \gamma$. Define

$$F.G(t,s) = \begin{cases} F(t,2s) & \text{if} \quad 0 \le s \le 1/2 \\ G(t,2s-1) & \text{if} \quad 1/2 \le s \le 1. \end{cases}$$

Note that $F.G$ is defined in two different manners on two closed subsets of $I^2 = I \times I$, which agree on the intersection of the two closed subsets (where $s = 1/2$). Thus $F.G$ is indeed a function $I^2 \to X$. It is evidently continuous when restricted to each of the two closed sets and hence continuous on all of I^2 by an application of the Gluing lemma. Now $F.G : \alpha \simeq \gamma$. This shows transitivity.

Finally, let $F(t,s) = \alpha(t)$. Then $F : \alpha \simeq \alpha$. This shows reflexivity. $\square$

We denote the equivalence class of $x \xrightarrow{\alpha} y$ by $\langle\alpha\rangle$ and sometimes write $x \xrightarrow{\langle\alpha\rangle} y$ to emphasize that any representative of $\langle\alpha\rangle$ is a path $x \to y$.

A subspace C of $\mathbb{R}^n$ is *convex* if all points on the ray $(1-t)p + tq$, $0 \le t \le 1$, are in C if p and q are. We note that in a such a space all paths $p \to q$ in C are homotopic. Indeed, let α and β be two such paths. Define $F(t,s) = (1-s)\alpha(t) + s\beta(t)$. The construction is possible by the linear structure of $\mathbb{R}^n$ and the image of F is in C by convexity. Then $F : \alpha \simeq \beta$.

Furthermore, if $f : X \to Y$ is a map and $F : \alpha \simeq \beta$ is a homotopy between paths in X, then $f \circ F$ is a homotopy $f \circ \alpha \simeq f \circ \beta$. This allows us to make the definition $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$.

Theorem 1.2 *Suppose $x \xrightarrow{\alpha,\alpha'} y \xrightarrow{\beta,\beta'} z$ are paths in a space X, that $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$. Then $\alpha.\beta \simeq \alpha'.\beta'$.*

Proof. Suppose $F : \alpha \simeq \alpha'$ and $G : \beta \simeq \beta'$. Define $F_*G : I^2 \to X$ by

$$F_*G(t,s) = \begin{cases} F(2t,s) & \text{if} \quad 0 \le t \le 1/2 \\ G(2t-1,s) & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

Continuity of F_*G follows as above by appealing to the Gluing lemma. Now $F_*G : \alpha.\alpha' \simeq \beta.\beta'$. $\square$

Using this we can safely define composition of $x \xrightarrow{\langle\alpha\rangle} y$ and $y \xrightarrow{\langle\beta\rangle} z$ as $\langle\alpha\rangle.\langle\beta\rangle = \langle\alpha.\beta\rangle$.

Theorem 1.3 *Suppose that $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} w$ are paths in X. Then*

1. *$\alpha.(\beta.\gamma) \simeq (\alpha.\beta).\gamma$, so that $\langle\alpha\rangle.(\langle\beta\rangle.\langle\gamma\rangle) = (\langle\alpha\rangle.\langle\beta\rangle).\langle\gamma\rangle$*
2. *$c_x.\alpha \simeq \alpha \simeq \alpha.c_y$, so that $\langle c_x\rangle.\langle\alpha\rangle = \langle\alpha\rangle = \langle\alpha\rangle.\langle c_y\rangle$,*
3. *$c_x \simeq \alpha.\alpha^{-1}$, so that $\langle c_x\rangle = \langle\alpha\rangle.\langle\alpha^{-1}\rangle$.*

Proof. Define a map $f : [0,\, 3] \to X$ by

$$f(t) = \begin{cases} \alpha(t) & \text{if} \quad 0 \le t \le 1 \\ \beta(t-1) & \text{if} \quad 1 \le t \le 2 \\ \gamma(t-2) & \text{if} \quad 2 \le t \le 3. \end{cases}$$

Continuity of f follows as usual. Define paths $0 \xrightarrow{\epsilon_1} 1$, $1 \xrightarrow{\epsilon_2} 2$ and $2 \xrightarrow{\epsilon_3} 3$ in the convex space $[0,\, 3]$, by $\epsilon_1(t) = t$, $\epsilon_2(t) = t+1$ and $\epsilon_1(t) = t+2$. Then $f \circ \epsilon_1 = \alpha$, $f \circ \epsilon_2 = \beta$ and $f \circ \epsilon_3 = \gamma$. Convexity of $[0,\, 3]$ gives $\epsilon_1.(\epsilon_2.\epsilon_3) \simeq (\epsilon_1.\epsilon_2).\epsilon_3$ since both are paths $0 \to 3$. By applying f to this we get $\alpha.(\beta.\gamma) \simeq (\alpha.\beta).\gamma$. This proves 1.

To prove 2 and 3, define $f : [0,\, 1] \to X$ by $f(t) = \alpha(t)$ (to avoid confusion). Then, as above, $f \circ \epsilon_1 = \alpha$, $f \circ \epsilon_1^{-1} = \alpha^{-1}$ while $f \circ c_0 = c_x$ and $f \circ c_1 = c_y$.

By convexity of $[0,1]$ we have $c_0.\epsilon_1 \simeq \epsilon_1 \simeq \epsilon_1.c_1$, since all three are paths $0 \to 1$. Applying f to this gives 2. Similarly, $c_0 \simeq \epsilon_1.\epsilon_1^{-1}$, since both are paths $0 \to 0$ in the convex space $[0,1]$. Applying f to this gives 3. $\square$

1.3 The fundamental group(s)

Let p be a point in a space X. We denote by $\pi_1(X,p)$ the set of homotopy classes of paths $p \to p$ in X. A path $p \xrightarrow{\alpha} p$ is called a loop in X based at p.

We can define a composition rule on this set by defining $\langle\alpha\rangle.\langle\beta\rangle = \langle\alpha.\beta\rangle$. From a theorem above we know that this is an associative composition and that $\langle c_p\rangle.\langle\alpha\rangle = \langle\alpha\rangle = \langle\alpha\rangle.\langle c_p\rangle$, so that the class of the constant loop at p is a unit element for the composition rule. We also know that $\langle\alpha\rangle.\langle\alpha^{-1}\rangle = \langle c_p\rangle$, so that $\langle\alpha^{-1}\rangle$ is an inverse of $\langle\alpha\rangle$ with respect to the composition, i.e. $\langle\alpha^{-1}\rangle = \langle\alpha\rangle^{-1}$.

With this we have given the set $\pi_1(X,p)$ the structure of a group. It is called *the fundamental group* of X based at p (the *base point*).

We check how this group depends on the choice of base point.

Theorem 1.4 *Let $p \stackrel{\omega}{\to} q$ be a path in X. Then the function $\omega_* : \pi_1(X,p) \to \pi_1(X,q)$ defined by $\omega_*(\langle\alpha\rangle) = \langle\omega^{-1}\rangle.\langle\alpha\rangle.\langle\omega\rangle$ is an isomorphism of groups.*

Proof. We need to check that $\omega_*(\langle\alpha\rangle.\langle\beta\rangle) = \omega_*(\langle\alpha\rangle).\omega_*(\langle\beta\rangle)$ and that $\omega_*\langle c_p\rangle = \langle c_q\rangle$, so that ω_* maps the unit element to the unit element.

But

$$\begin{aligned}
\omega_*(\langle\alpha\rangle.\langle\beta\rangle) &= \langle\omega^{-1}\rangle.\langle\alpha\rangle.\langle\beta\rangle.\langle\omega\rangle = \\
&= \langle\omega^{-1}\rangle.\langle\alpha\rangle.\langle c_p\rangle.\langle\beta\rangle.\langle\omega\rangle = \\
&= \langle\omega^{-1}\rangle.\langle\alpha\rangle.\langle\omega\rangle.\langle\omega^{-1}\rangle.\langle\beta\rangle.\langle\omega\rangle = \\
&= \omega_*(\langle\alpha\rangle).\omega_*(\langle\beta\rangle).
\end{aligned}$$

and

$$\omega_*(\langle c_p\rangle) = \langle\omega^{-1}\rangle.\langle c_p\rangle.\langle\omega\rangle = \langle\omega^{-1}\rangle.\langle\omega\rangle = \langle c_q\rangle.$$

Finally, the homomorphism $(\omega^{-1})_* : \pi_1(X,q) \to \pi_1(X,p)$ is inverse to ω_* so that this function is an isomorphism of groups. □

From this result it follows that *if a space is path-connected then any two fundamental groups of that space are isomorphic.*

Next we consider the effect of a map $f : X \to Y$ on fundamental groups.

Theorem 1.5 *If $f : X \to Y$ is a map, then the function $f_* : \pi_1(X,p) \to \pi_1(Y,f(p))$ defined by $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$ is a group homomorphism.*

Proof. We have

$$\begin{aligned}
f_*(\langle\alpha\rangle.\langle\beta\rangle) &= f_*(\langle\alpha.\beta\rangle) = \\
&= \langle f \circ (\alpha.\beta)\rangle = \langle (f\circ\alpha).(f\circ\beta)\rangle = \\
&= \langle f\circ\alpha\rangle.\langle f\circ\beta\rangle = f_*(\langle\alpha\rangle).f_*(\langle\beta\rangle),
\end{aligned}$$

and

$$f_*(\langle c_p\rangle) = \langle f \circ c_p\rangle = \langle c_{f(p)}\rangle.$$

□

It follows from this result that *if f is a homeomorphism then f_* is an isomorphism* with inverse given by $(f^{-1})_*$. Thus *homeomorphic spaces have isomorphic fundamental groups.*

A space X is *simply connected* if it is path-connected and $\pi_1(X,x)$ is a trivial group for some (hence any) $x \in X$.

2 The homotopy relation on maps and the fundamental group

In the last section we saw that spaces of the same homeomorphism type (i.e homeomorphic spaces) have isomorphic fundamental groups. In this section we introduce a cruder equivalence relation – being of the same homotopy type – on the class of all spaces and show that spaces of the same homotopy type have isomorphic fundamental groups.

Spaces of the same homotopy type can be wildly non-homeomorphic, but the fundamental group cannot see the difference between such spaces. This can be considered as a draw back if we are interested in determining if two spaces are homeomorphic or not, but it also has a positive side: it makes the fundamental group easier to calculate. Suppose for example that we would like to compute the fundamental group of a space X. If X turns out to be of the same homotopy type as a space Y of which we already know the fundamental group, then the fundamental group of X will be isomorphic to this group.

2.1 The homotopy relation

Two maps f, $g : X \to Y$ are *homotopic*, written $f \simeq g$, if there is a map $F : X \times I \to Y$, such that $F_0 = f$ and $F_1 = g$. Here $F_s(x) = F(x, s)$. We write such a map symbolically as $F : f \simeq g$, and say that F is a *homotopy* from f to g.

Theorem 2.1 *The relation $\simeq$ is an equivalence relation on the set of maps $X \to Y$.*

Proof. Given $f : X \to Y$ define $F(x, t) = f(x)$ for all $(x, t) \in X \times I$. Then $F : f \simeq f$, which shows reflexivity.

Given $F : f \simeq g$, define $F^{-1}(x, t) = F(x, 1 - t)$. Then $F^{-1} : g \simeq f$. This shows symmetry.

Given $F : f \simeq g$ and $G : g \simeq h$ define $F.G : X \times I \to Y$ as

$$F.G(x,t) = \begin{cases} F(x, 2t) & \text{if} \quad 0 \leq t \leq 1/2 \\ G(x, 2t-1) & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Thus $F.G$ is defined in two different ways on the closed subset $A = X \times [0, 1/2]$ and $B = X \times [1/2, 1]$ of $A \cup B = X \times I$. We note that the two definitions agree on $A \cap B = X \times \{1/2\}$ and consequently *do* define a function $X \times I \to Y$. Furthermore, $F.G$ is evidently continuous on A and B and hence on $A \cup B = X \times I$ by the Gluing lemma.

We now have $F.G : f \simeq h$. □

Next we check that the homotopy relation behaves well with respect to composition of maps:

Theorem 2.2 *Suppose that f and f' are homotopic maps $X \to Y$ and that g and g' are homotopic maps $Y \to Z$. Then $g \circ f \simeq g' \circ f'$*

Proof. Suppose, more specifically, that $F : f \simeq f'$ and $G : g \simeq g'$. Then $g \circ F : g \circ f \simeq g \circ f'$ while $G \circ (f' \times 1_I) : g \circ f' \simeq g' \circ f'$, so $g \circ f \simeq g' \circ f'$ by transitivity of the homotopy relation on maps $X \to Z$.□

A map $f : X \to Y$ is said to be a *homotopy equivalence* if there is a map $g : Y \to X$, called a homotopy inverse of f, such that

$$\begin{aligned} g \circ f &\simeq 1_X \\ f \circ g &\simeq 1_Y. \end{aligned}$$

Notice that f is then a homotopy inverse of g, so that g is also a homotopy equivalence. If f is a homeomorphism and we choose $g = f^{-1}$ we see that any homeomorphism is a homotopy equivalence.

Thus a homotopy equivalence behaves like a homeomorphism if you view it through homotopy glasses.

Two spaces X and Y are homotopy equivalent, or of the same *homotopy type*, if there is a homotopy equivalence $X \to Y$. Two spaces of the same homeomorphism type are also of the same homotopy type, so the latter is a coarser equivalence relation on spaces than the former.

2.2 Homotopy and the fundamental group

We next consider how homotopic map relates when we consider their effect on fundamental groups:

Theorem 2.3 *Suppose that f, $g : X \to Y$ are homotopic and $p \in X$ is a choice of base point. Then there is a path $f(p) \xrightarrow{\omega} g(p)$ in Y such that the composite $\omega_* \circ f_* : \pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\omega_*} \pi_1(Y, g(p))$ equals the map $g_* : \pi_1(X, p) \to \pi_1(Y, g(p))$.*

Recall that the map ω_* is an isomorphism of groups by a previous theorem. Thus, even though we cannot say that f_* and g_* are the same homomorphism they only differ by an isomorphism.

Proof. Suppose $F : f \simeq g$ and let $p \overset{\alpha}{\to} p$ be a loop at p in X. Define the map $H : I \times I \to Y$ to be the composite of $\alpha \times 1_I : I \times I \to X \times I$ with $F : X \times I \to Y$, so that $H(t, s) = F(\alpha(t), s)$. In $I \times I$ define four paths along the edges as follows:

$$\begin{array}{lcl} (0,0) \overset{\epsilon_1}{\to} (1,0) & \text{by} & \epsilon_1(t) = (t,0) \\ (0,1) \overset{\epsilon_2}{\to} (1,1) & \text{by} & \epsilon_2(t) = (t,1) \\ (0,0) \overset{\epsilon_3}{\to} (0,1) & \text{by} & \epsilon_3(t) = (0,t) \\ (1,0) \overset{\epsilon_4}{\to} (1,1) & \text{by} & \epsilon_4(t) = (1,t). \end{array}$$

Then, defining $\omega(t) = F(p, t)$ we have a path $f(p) \to g(p)$ and

$$\begin{array}{rcl} H \circ \epsilon_1(t) = F(\alpha(t), 0) & = & f \circ \alpha(t) \\ H \circ \epsilon_2(t) = F(\alpha(t), 1) & = & g \circ \alpha(t) \\ H \circ \epsilon_3(t) = F(\alpha(0), t) & = & F(p,t) = \omega(t) \\ H \circ \epsilon_4(t) = F(\alpha(1), t) & = & F(p,t) = \omega(t) \end{array}$$

Convexity of $I \times I \subset \mathbb{R}^2$ gives that $\epsilon_2 \simeq \epsilon_3^{-1}.(\epsilon_1.\epsilon_4)$, since both are paths $(1,0) \to (1,1)$ in $I \times I$.

Applying H to this gives

$$g \circ \alpha \simeq \omega^{-1}.((f \circ \alpha).\omega)$$

and

$$\begin{array}{rcl} g_*(\langle\alpha\rangle) & = & \langle g \circ \alpha \rangle = \\ & = & \langle \omega^{-1}.((f \circ \alpha).\omega) \rangle = \\ & = & \langle \omega^{-1} \rangle.\langle f \circ \alpha \rangle.\langle \omega \rangle = \\ & = & \omega_*(\langle f \circ \alpha \rangle) = \omega_* \circ f_*(\langle \alpha \rangle). \end{array}$$

Since α is arbitrary this shows the claim. $\square$

Finally, we show that spaces of the same homotopy type have isomorphic fundamental groups.

Theorem 2.4 *Suppose $f : X \to Y$ is a homotopy equivalence. Then $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ is an isomorphism for all choices of base points $p \in X$.*

Proof. Let $g : Y \to X$ be a homotopy inverse of f so that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. By the theorem above $(g \circ f)_* = g_* \circ f_* : \pi_1(X, p) \to \pi_1(Y, f(p)) \to \pi_1(X, g \circ f(p))$ deviates from $(1_X)_*$ (which is the identity homomorphism $\pi_1(X, p) \to \pi_1(X, p)$) by an isomorphism. Thus $g_* \circ f_*$ is an isomorphism and g_* has to be surjective and f_* has to be injective. Thus, since g is a homotopy equivalence too, g_* has to be injective. It follows that g_* is an isomorphism and consequently f_* is also an isomorphism. $\square$

3 Covering spaces and calculations

In this section we will consider one technique for calculating the fundamental group of a space. It is neither algorithmic nor applicable in all cases, but does lead to a calculation of the fundamental group of the Klein bottle for example.

The story starts with the following observation:

Suppose that G is a discrete group acting on a simply connected space X. Fix a base point $p \in X$ and let $q : X \to X/G$ denote the quotient map from X to the orbit space. Next, for each $g \in G$ choose a path $p \stackrel{\alpha_g}{\to} gp$. This is possible since X is path-connected and since X is even simply connected any two choices of paths $p \to gp$ are homotopic. We note that $q \circ \alpha$ is a loop in X/G based at $q(p)$, since $q(gp) = q(p)$.

This give us a function

$$\phi : G \to \pi_1(X/G, q(p)),\ \phi(g) = \langle q \circ \alpha_g \rangle = q_*(\langle \alpha_g \rangle).$$

This function is in fact a group homomorphism. Indeed, α_{gh} is a path $p \to (gh)p$, but so is the composite $p \stackrel{\alpha_g}{\to} gp \stackrel{g\alpha_h}{\to} g(hp)$. Here $g\alpha_h$ is the path $t \mapsto g(\alpha_h(t))$. Since X is assumed to be simply connected the two paths are homotopic, so $\langle \alpha_{gh} \rangle = \langle \alpha_g . g\alpha_h \rangle$. Also, note that $q \circ (g\alpha_h) = q \circ \alpha_h$, since $g(\alpha_h(t))$ and $\alpha_h(t)$ are in the same orbit of the action.

This leads to

$$\phi(gh) \quad = \quad q_*(\langle \alpha_{gh} \rangle) = q_*(\langle \alpha_g . g\alpha_h \rangle) = q_*(\langle \alpha_g \rangle).q_*(\langle g\alpha_h \rangle) = q_*(\langle \alpha_g \rangle).q_*(\langle \alpha_h \rangle) = \phi(g).\phi(h).$$

Thus ϕ respects the composition rule. The other requirement of a group homomorphism is that it maps the unit element 1 of G to the unit element $\langle c_{q(p)} \rangle$ of $\pi_1(X/G, q(p))$. We check this:

$$\phi(1) = q_*(\langle \alpha_1 \rangle) = q_*(\langle c_p \rangle) = \langle c_{q(p)} \rangle.$$

We have used that both α_1 and c_p are paths $p \to 1p = p$ in X.

3.1 Covering actions

We cannot generally expect the homomorphism ϕ (of the previous section) to be injective, let alone an isomorphism. Suppose for example that $gp = p$ for some $g \neq 1$ in G, then $\phi(g) = \langle c_{q(p)} \rangle$. A minimal requirement for ϕ to be injective is that the action of G on X is *free*: $gx = x$ for some $x \in X$ only if $g = 1$. If G is finite and X is Hausdorff, this turns out to be the only thing we need to assume to make ϕ an isomorphism.

In general we have to assume that the action is a covering action (a term not used in Armstrong's book):

Definition 3.1 *A group action $G \times X \to X$ is a* covering action *if any point in X has an open neighborhood U such that $GU = \coprod_{g \in G} gU$.*

Here are three general types of actions which are covering actions

1. If H is a discrete subgroup of a topological group G, then the action $H \times G \to G$, $(h, g) \mapsto gh$ (multiplication in G) is a covering action.

   To verify the condition let first $p = 1$ the unit element of G (and H). Since H is discrete there is an open set O in G such that $O \cap H = \{1\}$. Continuity of the map $m : G \times G \to G$, $m(x, y) = xy^{-1}$ and the fact that $(1, 1) \in m^{-1}(O)$ allows us to find a basic open set $V \times V \subset m^{-1}(O)$ containing $(1, 1)$. The image of $V \times V$ under m is $V \cdot V^{-1} \subset O$ and contains 1.

   Suppose that $x \in hV \cap h'V$, where h, $h' \in H$. Then $x = hv = h'v'$, some v, $v' \in V$. This gives $h^{-1}h' = v(v')^{-1} \in H \cap O = \{1\}$, so $h = h'$ and hV and $h'V$ are thus disjoint unless $h = h'$.

   If $p = x$ is a general point of G then Vx is an open set containing x and $hVx \cap h'Vx = \emptyset$, unless $h = h'$.

2. Suppose G is finite and acts freely on a Hausdorff space X, then the action is a covering action.

   To verify this, fix $p \in X$ and $g \in G$, $g \neq 1$. Then $p \neq gp$ and since X is Hausdorff we can find open disjoint sets O^g containing p and O_g containing gp. Then $U^g = O^g \cap g^{-1}O_g$ is an open set containing p such that U^g and gU^g are disjoint: the first is contained in O^g the second one in O_g and these two are disjoint.

   Next put $U = \bigcap_{g \in G} U^g$ and suppose $gU \cap g'U \neq \emptyset$. Multiplying by g^{-1} then gives $U \cap g^{-1}g'U \neq \emptyset$. But $U \subset U^{g^{-1}g'}$ and $g^{-1}g'U \subset g^{-1}g'U^{g^{-1}g'}$ where $U^{g^{-1}g'}$ and $g^{-1}g'U^{g^{-1}g'}$ are disjoint unless $g^{-1}g' = 1$, i.e. $g = g'$.

3. Suppose G acts freely on a metric space X through isometries, i.e. multiplication by $g \in G$ preserves distances: $d(gx, gy) = d(x, y)$. If, in addition, each point p has a neighborhood O such that $O \cap Gp = \{p\}$, (that is p is open in Gp)then the action is a covering action.

   To verify this choose $\epsilon > 0$ such that $B_\epsilon(p) \subset O$. Then $d(gp, p) > \epsilon$ when $g \neq 1$ and $U = B_{\epsilon/2}(p)$ has the required property. Indeed, $gB_{\epsilon/2}(p) = B_{\epsilon/2}(gp)$ and multiplying $gB_{\epsilon/2}(p) \cap hB_{\epsilon/2}(p)$ by g^{-1} gives the homeomorphic $B_{\epsilon/2}(p) \cap B_{\epsilon/2}(g^{-1}hp)$. But this set is empty if $g^{-1}h \neq 1$ since then $d(p, g^{-1}hp) > \epsilon$.

Here are three examples of the situations above:

1. $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$ (which is a topological group with respect to ordinary addition). Here the orbit space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1 through the exponential map.

   As $\mathbb{R}$ is simply connected we have the homomorphism $\phi : \mathbb{Z} \to \pi_1(S^1, 1)$. To spell out the details of this map we take $p = 0$, and we choose α_n to be the linear path $0 \to n$ defined by $\alpha_n(t) = tn$, which then maps to the loop $\exp(tn2\pi i)$ based at 1 in S^1, i.e n full revolutions on the circle.

2. Let C_2 be the cyclic group of order 2 (written multiplicatively) with generator τ. A free action of this finite group on the Hausdorff space S^n is defined by $\tau \cdot x = -x$. Here the orbit space is the projective space $\mathbb{P}^n$ of dimension n.

   We will see that S^n is simply connected if $n \geq 2$. With this proviso, and taking $p = \mathbf{e}_{n+1}$, the north-pole, we have the homomorphism $\phi : C_2 \to \pi_1(\mathbb{P}^n, [\mathbf{e}_{n+1}])$.

   To spell out the definition of ϕ we choose a path from $\mathbf{e}_{n+1}$ to $\tau\mathbf{e}_{n+1} = -\mathbf{e}_{n+1}$, for example $\alpha(t) = \cos(t\pi)\mathbf{e}_1 + \sin(t\pi)\mathbf{e}_{n+1}$. When this path is transported to $\mathbb{P}^n$ using the quotient map it becomes a loop at $[\mathbf{e}_{n+1}]$.

3. Define A and B to be the isometries $A(x, y) = (x, y + 1)$ and $B(x, y) = (x + 1, 1 - y)$ of the metric space $\mathbb{R}^2$. They generate a subgroup of the group of all homeomorphisms of $\mathbb{R}^2$, which we now describe.

   We notice that $ABA(x, y) = AB(x, y + 1) = A(x + 1, -y) = (x + 1, 1 - y) = B(x, y)$, so that $ABA = B$ or $BA = A^{-1}B$. Hence we can move an A passed a B at the expense of inverting it. It follows that $A^nB^mA^rB^s = A^{n+(-1)^m r}B^{m+s}$. We next show that $A^nB^m = A^rB^s$ only if $n = r$ and $m = s$ by calculating

   $$A^nB^m(x, y) = \begin{cases} A^n(x + m, y) \\ A^n(x + m, 1 - y) \end{cases} = \begin{cases} (x + m, y + n) & \text{if} \quad m \text{ is even} \\ (x + m, n + 1 - y) & \text{if} \quad m \text{ is odd.} \end{cases}$$

   Thus the group G generated by A and B is $G = \{A^nB^m | n, m \in \mathbb{Z}\}$ and multiplication is performed according to the rule $A^nB^mA^rB^s = A^{n+(-1)^m r}B^{m+s}$ and the action on $\mathbb{R}^2$ is as described above. Notice that the action is free! As the distance between (x, y) and $A^nB^m(x, y)$, with $(n, m) \neq (0, 0)$, is $\sqrt{m^2 + n^2} \geq 1$ if m is even and $\sqrt{m^2 + (n + 1 - 2y)^2} \geq 1$ if m is odd the condition in 3 above is satisfied.

   It's not to hard to convince one self that all orbits of this action passes trough I^2 exactly once, with the exceptions that $(0, y)$ is in the same orbit as $(1, 1 - y)$ and $(x, 0)$ is in the same orbit as $(x, 1)$. The quotient space $\mathbb{R}^2/G$ is homeomorphic to the Klein-bottle K.

   As $\mathbb{R}^2$ is simply connected we have a homomorphism $\phi : G \to \pi_1(K, [0, 0])$.

3.2 Covering projections

We now return to definition 3.1 and analyze its implications on the (open) quotient map $q : X \to X/G$. Let U be an open set containing x as in the definition of a covering action. Then $V = q(U)$ is an open neighborhood of Gx in X/G such that $q^{-1}(V)$ is a disjoint union, namely $\coprod_{g \in G} gU$ of open sets in X. Furthermore, the restriction of q to each of these open sets is an open bijection $q|gU \to V = q(U)$ and hence a homeomorphism.

To focus on the relevant information in this situation, we formalize this as follows. Suppose $p : X \to Y$ is a surjective map. An open subset V of Y is *evenly covered* by p if $p^{-1}(V)$ is a disjoint union $\coprod_{j \in J} U_j$, such that the restriction of p to each of the U_j:s is a homeomorphism $p| : U_j \overset{\cong}{\to} V$. It follows from this that the fiber $p^{-1}(y)$ is a discrete subspace of X consisting of one point in each of the disjoint open sets U_j, $j \in J$.

We next define a surjective map $p : X \to Y$ to be a *covering projection* (X a covering space of Y, Y is the base space of p) if each point of Y has an open neighborhood evenly covered by p. The prime example of covering projection is the case when p is the quotient map $X \to X/G$ of a covering action of a group G on X. (Actually, pretty much all covering projections arise in this way, but that's another story.)

We now consider how covering projections relates to homotopy classes of paths in Y and X. We do this by proving a sequence of lemmas.

A *lift* of a map $f : Z \to Y$ with respect to $p : X \to Y$ is a map $\tilde{f} : Z \to X$, such that $f = p \circ \tilde{f}$. It's also called a *factorization* of f through p.

Lemma 3.1 *Suppose Z is connected and $p : X \to Y$ is a covering projection. If $f : Z \to Y$ is a map with image contained in an open set of Y evenly covered by p and $x \in X$ and $z \in Z$ are given points with $f(z) = p(x)$, then there is a unique lift $\tilde{f}$ of f with respect to p with $\tilde{f}(z) = x$.*

Proof. By the assumptions we have $f : Z \to V$ where $p^{-1}(V) = \coprod_j U_j$, where each U_j is open in X. Suppose $x \in U_{j_0}$. We have the homeomorphism $q| : U_{j_0} \to V$. Observe that the connectivity of Z means that $\tilde{f}(Z)$ has to be contained in U_{j_0}, which actually determines $\tilde{f}$ as $\tilde{f} = (q|)^{-1} \circ f$ (with $\tilde{f}(z) = x$). □

Lemma 3.2 *Suppose $p : X \to Y$ is a covering projection and x is a fixed point of X. Then any map $F : I^2 \to Y$ has a unique lift $\tilde{F} : I^2 \to X$ with $\tilde{F}(0, 0) = x$.*

Proof. By assumption Y can be covered by open sets evenly covered by p. The inverse images of these under F form an open cover of I^2, which is a compact metric space. Consequently, by the Lebesgue's lemma, we can choose equidistant points on the x- and y-axises of I^2 so that each of the resulting small squares in I^2 is mapped into an evenly covered open subset of Y by F.

Starting with the lower left hand square in this subdivision, there is, by the lemma above, a unique lifting of F defined on this square with $(0, 0) \mapsto x$. Moving to the next square on the right side of it we can extend the lift uniquely to this square, again using the lemma, which further assures that the lifts agrees on the line segment common for the two squares. Continuing in this fashion we can find a unique lift of F on all of the squares of the bottom row in the subdivision of I^2. We next move to the second row from the bottom in the subdivision and find that we can repeat the argument to extend the partially defined lift uniquely to include the second row,too. Repeating row by row we conclude that the unique lift $\tilde{F}$ with $\tilde{F}(0, 0) = x$ exists. □

Lemma 3.3 *Suppose $p : X \to Y$ is a covering projection and x is a fixed point of X. Then any map $\alpha : I \to Y$ has a unique lift $\tilde{\alpha} : I \to X$ with $\tilde{\alpha}(0) = x$.*

Proof. Similar to, but easier than, the proof of lemma 3.2. □

(You can actually get lemma 3.3 from lemma 3.2 by considering the map $I^2 \to Y$, $(t, s) \mapsto \alpha(s)$. Details are left to the reader.)

3.3 Covering projections and paths

Using the lemmas of the previous section we now draw the following conclusions on the relation between paths in Y and paths in X in a covering projection:

Theorem 3.1 *Suppose $p : X \to Y$ is a covering projection and $x \in X$ a fixed point. Then*

1. *any path α in Y starting at $p(x)$ in the image of a unique path α in X starting at x*

2. *if $\alpha \simeq \beta$ and $\tilde{\beta}$ is a lift of β starting at x, then $\tilde{\alpha} \simeq \tilde{\beta}$.*

Proof. In view of lemma 3.3 only the second part requires an argument.

Let $F : \alpha \simeq \beta$. Then F has, by lemma 3.2, a unique lift $\tilde{F}$ with $\tilde{F}(0,0) = x$.

To see that $\tilde{F}$ is a homotopy $\tilde{\alpha} \simeq \tilde{\beta}$ we observe first that $p \circ \tilde{F}(0,s) = F(0,s) = p(x)$ and $p \circ \tilde{F}(1,s) = F(1,s) = \alpha(1)$, so that $\tilde{F}(0,s)$ and $\tilde{F}(1,s)$ are lifts of constant paths. By uniqueness they are constant too. Thus $\tilde{F}$ is a path homotopy between paths in X starting at x.

Now $p \circ \tilde{F}(t,0) = F(t,0) = \alpha(t)$ and $p \circ \tilde{F}(t,1) = F(t,1) = \beta(t)$. Uniqueness implies that $\tilde{F}(t,0) = \tilde{\alpha}(t)$ and $\tilde{F}(t,1) = \tilde{\beta}(t)$. □

3.4 The conclusion

We finally return to the starting point of the story:

Theorem 3.2 *Suppose G acts on a simply connected space X by a covering action. Then, for any $x \in X$ the map*

$$\phi : G \to \pi_1(X/G, q(x))$$

defined above is an isomorphism.

Proof. We first show surjectivity. Let $\langle\beta\rangle$ be in $\pi_1(X/G, q(x))$ represented by a loop $q(x) \xrightarrow{\beta} q(x)$. Let $\tilde{\beta}$ be a lift of β starting at x. Then $p \circ \tilde{\beta}(1) = \beta(1) = p(x)$, so $\tilde{\beta}(1) = gx$, some $g \in G$. Since X is simply connected $\alpha_g \simeq \tilde{\beta}$, since both are paths $x \to gx$, and

$$\phi(g) \quad = \quad q_*(\langle\alpha_g\rangle) = q_*(\langle\tilde{\beta}\rangle) = \langle\beta\rangle.$$

Next, to show injectivity of a group homomorphism it suffices to show that only the unit element is mapped to the unit element. So suppose $\phi(g) = \langle c_{q(x)}\rangle$, i.e $q \circ \alpha_g \simeq c_{q(x)}$. Then α_g is a lift of $q \circ \alpha_g$ while c_x is a lift of $c_{q(x)}$. By 2 of Theorem 3.1 we have that $\alpha_g \simeq c_x$, in particular $gx = \alpha_g(1) = c_x(1) = x$. By the condition on the action this implies that $g = 1$. □

This theorem gives the following (non-algorithmic) method of trying to compute the fundamental group of a space Y:

> Find a simply connected space X and a covering action of a discrete group G on X such that X/G is homeomorphic (or even less: homotopy equivalent) to Y.

The examples 1 − 3 above leads to the computations (once it's know that S^n is simply connected for $n \geq 2$)

1. $\pi_1(S^1) \cong \mathbb{Z}$ and a generator of $\pi_1(S^1, 1)$ is $\langle\exp(2\pi it)\rangle$.
2. $\pi_1(\mathbb{P}^n) \cong C_2$, when $n \geq 2$ and a generator in $\pi_1(P^n)$ is $\langle q(\cos(\pi t)\mathbf{e}_1 + \sin(\pi t)\mathbf{e}_{n+1})\rangle$.
3. $\pi_1(K) \cong G$, where $G = \{A^n B^m \mid n, m \in \mathbb{Z}\}$ and $A^n B^m A^r B^s = A^{n+(-1)^m r} B^{m+s}$. The elements A and B corresponds to the classes of loops in K that you get from the paths $a(t) = (t, 0)$ and $b(t) = (t, t)$ in I^2 by applying the quotient map from I^2 to K.

3.5 The fundamental group of a topological group

We have seen that the fundamental group of S^1 is abelian: the multiplication is commutative. This is no coincidence since the fundamental group of (any path-component) of a topological group turns out to be abelian. We show this.

First note that if $1 \xrightarrow{\alpha,\beta} 1$ are loops in a topological group G, then there we could use the multiplication in the group to get a second way of composing them. Define $\alpha * \beta(t) = \alpha(t) \cdot \beta(t)$ where the dot denotes multiplication in G. We will also show that $\langle\alpha * \beta\rangle = \langle\alpha\rangle.\langle\beta\rangle$.

Theorem 3.3 *Suppose that G is a topological group and let $1 \overset{\alpha,\beta}{\to} 1$ be loops at the unit element of G. Then*

$$\langle\alpha\rangle.\langle\beta\rangle = \langle\alpha * \beta\rangle = \langle\beta\rangle.\langle\alpha\rangle.$$

Thus the fundamental group of a topological group is abelian.

Proof. Define a map $F : I^2 \to G$ by $F(t,s) = \alpha(t) \cdot \beta(s)$. In I^2 let $(0,0) \overset{\epsilon_1}{\to} (1,0)$, $(0,1) \overset{\epsilon_2}{\to} (1,1)$, $(0,0) \overset{\epsilon_3}{\to} (0,1)$, $(1,0) \overset{\epsilon_4}{\to} (1,1)$ be the standard linear paths. Also let $(0,0) \overset{\epsilon_5}{\to} (1,1)$ be the linear diagonal path. Applying F to these paths gives in turn, α, β, β, α and $\alpha * \beta$.

Since I^2 is a convex subset of $\mathbb{R}^2$ we have

$$\epsilon_1.\epsilon_4 \simeq \epsilon_5 \simeq \epsilon_3.\epsilon_2,$$

since all three are paths $(0,0) \to (1,1)$. Applying F to this gives

$$\alpha.\beta \simeq \alpha * \beta \simeq \beta.\alpha$$

□

Example. Since we have seen that the fundamental group of K is non-abelian it's not possible to find a continuous multiplication on it which makes it into a group.

Example Suppose G is a simply connected topological group (such as S^3 for example), then any normal discrete subgroup H has to be abelian. In fact, normality of H ensures that G/H is a topological group, so $H \cong \pi_1(G/H)$ has to be abelian.

Example The subgroup $Q_8 = \{\pm 1,\ \pm i,\ \pm j,\ \pm ij\}$ of S^3 is not normal. (Try to prove this directly, it's not hard.)