Groups Actions: Part 1 (general set up) by J. Steif, January, 2019

There are tons of lecture notes on group actions on the web, for example, search for the very nice notes by Keith Conrad. Here I just, very briefly, present the most elementary basics. (Let me know any corrections you find!).

Definition: If G is a group and X is a set, a **group action** of G on X is a mapping

$$f : G \times X \to X$$

satisfying

$$(1)\ f(g_1g_2, x) = f(g_1 f(g_2, x))\ \ \forall g_1, g_2 \in G, x \in X$$

and

$$(2)\ f(e, x) = x\ \ \forall x \in X$$

where e is the identity in G.

Remarks. (i). It is simplest to abbreviate $f(g, x)$ by (g, x) which we now do.
(ii). You should think of (g, x) as "the result obtained when g acts on x". So "g takes x to another element of X" which we call (g, x).
(iii). The key equality (1) says that if you first let g_2 act on x and then act on the result by g_1, you end up with the same thing as you would if you had first multiplied g_1 and g_2 in the group and then let the product act on x. (It has a very similar flavor, but not exactly the same thing, as the definition of a homomorphism of groups.)

Exercise: Show that if you fix g, then the mapping from X to X given by x goes to (g, x) is a bijection. Is condition (2) in the definition of a group action superfluous?

Example 1. Z acts on $Z/12$ by $(n, x) = x + n$. Verify that this is a group action. What is happening geometrically? If $g = 24$, what is the corresponding bijection of $Z/12$?

Example 2. $Z/4$ acts on $Z/8$ by $(g, x) = 2g + x$. Verify that this is a group action.

Exercise. Show that a group action of G on X is equivalent to a group homomorphism from G to S_X where the latter is the symmetric group on X. (Hint. Use the previous exercise).

Group actions (applied to the correct objects) are the nicest way (in my opinion) to prove all of the Sylow theorems from group theory.

We now discuss the various key players that arise when studying group actions.

Definition. Given a group action, we say $x \sim y$ if there exists g such that $(g, x) = y$ (i.e., if some g takes x to y).

Exercise. Show that $\sim$ is an equivalence relation.

Definition. Given a group action, the equivalence classes for the above equivalence relation are called **orbits**.

Definition. A group action is called **transitive** if there is only one orbit, meaning you can get from any x to any y using some element of g.

Definition. Given a group action of G on X and given $x \in X$, the **stabilizer** of x, denoted S_x, is $\{g : (g, x) = x\}$. These are the group elements which send x to itself, hence the word stabilizer.

Exercise. Show that S_x is a subgroup of G.

Exercise. If x and y are in the same orbit, then S_x and S_y are conjugate subgroups; i.e. there exists $g \in G$ so that $gS_xg^{-1} = S_y$. (Hint. If g takes x to y, then verify $S_x = g^{-1}S_yg$.)

Lemma. If G is finite and $x \in X$, then the orbit of x, denoted by O_x, is finite and satisfies

$$|O_x| = [G : S_x]$$

where the latter denotes the index of the subgroup S_x in G.

Outline of Proof. (Fill in details if needed). Map the left cosets of S_x, denoted by G/S_x, to O_x by sending gS_x to (g, x). You need to verify this map is well-defined and a bijection. Here is why it is well defined. We need to show that $g_1 S_x = g_2 S_x$ implies $(g_1, x) = (g_2, x)$. But the first equality says that $g_2^{-1} g_1$ is in S_x which implies $(g_2^{-1} g_1, x) = x$. Now act by g_2 on both sides to obtain $(g_1, x) = (g_2, x)$. Check the other details. QED

Corollary. If G is a finite group, then for all x, we have $|O_x|$ divides $|G|$.

Definition. For $g \in G$, let F_g be $\{x : (g, x) = x\}$. This is the set of x fixed by g. (Think of the similarity and difference between this definition and that of a stabilizer).

Definition. An action is called **fixed point free** if for all $g \neq e$, we have $F_g = \emptyset$. This means that every element of g except for the identity ($F_e = X$ of course) moves *every* x in X.

Further exercise. Show that $\cap_x S_x$ is the kernel of the homomorphism from G to S_X which represents the group action.

One further topic. **Burnside Polya counting**. This is just an aside. Read if you want.

Motiviation. Here is a combinatorial problem. You have a pizza with 12 slices and each slice can be colored in 2 colors. How many ways can you do it? Of course 2^{12}. But now I say that if you can "rotate" one pizza into another, I don't want to consider them different. So, for example, there is only 1 pizza with 1 black slice and only 1 pizza with 2 black slices which are separated by 4 (of course 4 can be any number here). Now, I ask again how many different pizzas there are. I haven't thought so much about it but I would guess that, without any theory, solving it would take a good amount of time. Here is a way to count it fairly quickly with some theory.

Theorem. Let the finite group G act on the finite set X. Then the number of orbits is

$$\frac{1}{|G|} \sum_{g \in G} |F_g|.$$

Proof. Consider the set $S = \{(g, x) \in G \times X : gx = x\}$. Note, I changed notation now. Here (g, x) actually means the pair (g, x) and $gx = x$ means g takes x to x. (The latter should be written $(g, x) = x$ but that would clearly cause some confusion). In words, S is the set of pairs g and x for which g fixes x.

We now compute the size of S in two ways, first summing over x and then g and then summing in the other order. This immediately leads to

$$\sum_{g \in G} |F_g| = \sum_{x \in X} |S_x|.$$

Dividing by $|G|$, we get

$$\frac{1}{|G|} \sum_{g \in G} |F_g| = \sum_{x \in X} \frac{|S_x|}{|G|} = \sum_{x \in X} \frac{1}{[G : |S_x|]} = \sum_{x \in X} \frac{1}{|O_x|}$$

where we used a previous lemma for the last step. Looking at the last expression, what is the contribution to this sum when I sum over x's in some fixed orbit? It is exactly 1 and so the final sum is the number of orbits. QED

Coming back to the pizza problem, we let X be the set of all 2^{12} colorings of the pizza. We let the group $Z/12$ act on X by rotations. The original question now comes down to asking what the number of orbits is for this group action.

Using the above theorem, we get

$$\frac{1}{12}\sum_{i=0}^{11}|F_i|.$$

It is not so hard to show (verify!) that the numbers $|F_0|, |F_1|, \ldots, |F_{11}|$ are $2^{12}, 2, 4, 8, 16, 2, 64, 2, 16, 8, 4, 2$. This gives 352 orbits, much smaller than the original $2^{12} = 4096$ number of pizzas.