Groups Actions: Part 2 (Back to topology) by J. Steif, January, 2019

(Let me know any corrections you find!).
We now return to topology. Let G be a finite or countable group.

Definition: A "G-action on X by homeomorphisms" is a group action of G on X such that for all $g \in G$, the bijection x goes to (g, x) is a homeomorphism of X.

Example. Z^2 acts on R^2 by translations, $((a, b), (x, y)) = (a + x, b + y)$. Verify it is a group action by homeomorphisms.

Definition: If we have a G-action on X by homeomorphisms, we can partition X into orbits (as before, x and y are in the same orbit if there is $g \in G$ with $(g, x) = y$). The **orbit space**, denoted by X/G, is the set of orbits endowed with the quotient topology obtained from the canonical map from X to X/G.

Exercise. Check what the orbits are in the previous example and convince yourself that the orbit space is the 2-d torus.

Note that the fundamental group of the torus is Z^2, which is exactly what the original group was. Coincidence? NO. R^2 being simply connected was important however.

This leads to the following **hoped for** generalization.

Imprecise Theorem. If G is a countable group acting by homeomorphisms on a simply connected space X, then **under certain conditions**, $\pi_1(X/G)$ is isomorphic to G.

The following shows why some conditions are needed. Let G be the rational numbers Q and let them act on R by translation. This of course gives a group action by homeomorphisms (verify).

Exercise. Show that the quotient topology on the orbit space $X/G = R/Q$ is in fact the indiscrete topology (the only open sets are the whole space and the empty set). Although it is strange to talk about the fundamental group in such cases, show that the fundamental group is trivial. In particular, it is not isomorphic to the original group acting which was Q.

Remark. While the above example is nice in that one sees how bad quotient spaces can be, if we just want an example where the imprecise theorem fails without some conditions, one could simply let Z^3 act on R^2 where the last coordinate of the group element "doesn't do anything" meaning $((a, b, c), (x, y)) = (a + x, b + y)$. Verfiy this is still a group action by homeomorphims whose quotient space is (still) the 2-d torus and so its fundamental group, Z^2, is certainly not the original group Z^3.

The following gives the key property needed to have a general theorem and in particular rules out the situation in the previous remark, as well as the example of the rationals acting.

Definition. If G is a countable group acting by homeomorphisms on a space X, we say it acts **properly discontinuously** if for all $x \in X$, there is a neighbourhood U of X such that for all $g \neq e$, $g(U) \cap U = \emptyset$.

Exercise. Show that this implies $g_1(U) \cap g_2(U) = \emptyset$ for all $g_1 \neq g_2$. Show that the action of Q on R given above is not properly discontinuous. Show that the action above of Z^2 on R^2 is properly discontinuous but the action, above, of Z^3 on R^2 is not.

We can now state the real theorem.

Theorem. If G is a countable group acting by homeomorphisms on a simply connected space X and is acting properly discontinuously, then $\pi_1(X/G)$ is isomorphic to G.

Proof. We do this in class. Here are the key steps.

Step 1. Show that the mapping X to X/G (the latter with the quotient topology) is an open map.

Step 2. Show that the above mapping is a covering map.

Step 3. We define a map from G to $\pi_1(X/G)$. Then one needs to show this is an isomorphism of groups.

Fix $x_0 \in X$. Given $g \in G$, choose a path γ_g from x_0 to gx_0. Consider the loop in X/G from $\pi(x_0)$ to itself given by $\pi(\gamma_g)$. We send g to $[\pi(\gamma_g)]$.

One needs to show this is well-defined. This is basically because if we have to different paths in X from x_0 to gx_0, γ_g and γ_g', then these are homotopic since X is simply connected, and this homotopy can be pushed down, via π, to a homotopy between $\pi(\gamma_g)$ and $\pi(\gamma_g')$.

One has to show bijective which is not too hard. And then one has to show it is actually a group homomorphism, which is a little harder. QED (sort of).