Fundamental group of the Klein bottle as a semi-direct product by J. Steif, January, 2019

Based on the last three sets of notes, we give an outline of the derivation for the fundamental group of the Klein bottle.

Consider the group G of homeomorphisms of R^2 generated by t and u where

$$t(x, y) = (x + 1, y)$$

and

$$u(x, y) = (-x + 1, y + 1)$$

We first outline the proof that the fundamental group of the Klein bottle is G above. This would follow from "groups actions, Part 2" if we show that the action is properly discontinuous and that R^2/G is homeomorphic to the Klein bottle.

t and u satisfy a certain relation which is the key to everything.

Claim $tutu^{-1} = e$.

Proof. Compute. QED

A consequence of this is that $ut = t^{-1}u$, $u^{-1}t = t^{-1}u^{-1}$, $ut^{-1} = tu$ and $u^{-1}t^{-1} = tu^{-1}$. Check this.

As a result, every element of G can be expressed as $t^k u^\ell$ with $k, \ell \in Z$ since we can reach such an expression from any element of G by "pushing the t's to the left of the u's adjusting powers where necessary".

Proof of being properly discontinuous. Given x, let U be the open ball of radius $1/10$ around x. We need to show that $t^k u^\ell(U) \cap U = \emptyset$ if k or ℓ differ from 0. But if k differs from 0, then we have moved at least 1 unit in the y direction and if $k = 0$, then $\ell \neq 0$ and we have moved at least 1 unit in the x direction. QED

Now we consider R^2/G. It is clear that every orbit has a representative in $[0, 1]^2$. Check this. It is clear that no two elements in $(0, 1)^2$ are in the same orbit since we are always moving by 1. What happens on the boundary? This is the crucial part. One sees that the left and right sides are identified "both going upwards". The top and bottom are also identified but now the top moving left to right is identified to the bottom when the bottom is moving right to left. This by definition is the Klein bottle.

The above all shows that R^2/G is homeomorphic to the Klein bottle and hence the fundamental group of the Klein bottle is G.

Lastly, we want to show that G is isomorphic to the nontrivial semi-direct product (discussed in the third set of notes) of Z with itself, denoted by $Z \times_\phi Z$, with multiplication given by

$$(n, m)(n', m') = (n + (-1)^m n', mm').$$

Consider the map from $Z \times_\phi Z$ to G given by

$$(k, \ell) \to t^k u^\ell.$$

By what we had done above, this set mapping is onto and it is not hard to see it is injective. We show it is a group homomorphism. Given (k, ℓ) and (k', ℓ') if we first multiply in the semi-direct product and then map to G we get

$$t^{k+(-1)^\ell k'} u^{\ell+\ell'}.$$

If on the other hand, we first map to G and then multiply in G, we get $t^k u^\ell t^{k'} u^{\ell'}$. To show these are equal, we need that

$$t^{(-1)^\ell k'} u^\ell = u^\ell t^{k'}.$$

But our basic relation and its consequences immediately gives us this. Note that each time we pass a t through a u, the power of u stays the same and the power of t is multiplied by -1. Each t has to be pushed through ℓ copies of u, each time the power being multiplied by -1. Hence its final power is given by $(-1)^\ell$. QED

We finally end this by describing the fundamental group in a completely different way, namely in terms of generators and relations. These are discussed in the book. The final thing is that the fundamental group of the Klein bottle is also given by the group presentation $(t, u : tutu^{-1})$. This is, by definition, the quotient group of the free nonabelian group on the two letters t and u obtained by quotiening out by the normal subgroup generated by the element $tutu^{-1}$.