Mathematical Sciences (MV), GU
March 18, 2019, 14:00-18:00.

No aids (closed book, closed notes).
Phone: Jeffrey Steif 070-2298318
Presence of teacher: $\sim 15:00$ and $\sim 16:30$

Exam in MMA 100 Topology, 7.5 HEC.

8 problems. 24 points= 8×3. Grade limits: 12p for Godkänd (Pass). 18p for Väl Godkänd (Very Good). Each problem is graded as a whole; so it is not necessarily the case that if a problem has 4 parts then each part is worth $3/4$ of a point.

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1. In the following X and Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$.

   [a] Consider the statement $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Prove or give a counterexample.

   [b] Consider the statement $\overline{A \cap B} = \overline{A} \cap \overline{B}$. Prove or give a counterexample.

   [c] Show that if C and D are closed sets in X and Y respectively, then $C \times D$ is closed in $X \times Y$.

   [d] Consider the statement $\overline{A \times B} = \overline{A} \times \overline{B}$. Prove or give a counterexample.

2. **[a]** Explain how a metric space gives rise to a topology.

   [b] Assume that X and Y are metric spaces which are homeomorphic as topological spaces and assume that X is a complete metric space. Is Y necessarily then a complete metric space? Prove or give a counterexample.

   [c] Give an example of a topological space for which each 1 point set is closed but such that the topology does not arise from a metric; i.e., the space is not metrizable. Briefly justify the claim that your example works.

3. **[a]** Prove that a compact subspace of a Hausdorff space is closed.

   [b] Give a counterexample to **[a]** if the Hausdorff assumption is removed.

   [c] Assuming **[a]**, prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

   [d] Show that **[c]** is false if the compactness assumption is removed.

4. Consider the set $[0,1]^N$ both in the product topology and in the uniform topology (the latter one is the topology generated by the metric $d(x,y) = \sup_i |x_i - y_i|$). N denotes the positive integers.

   [a] Is $[0,1]^N$ in the product topology metrizable? If so, describe a metric which generates it. Give a very brief explanation of why it works.

   [b] Is $[0,1]^N$ in the uniform topology compact? Justify your answer.

   [c] Is the metric space $[0,1]^N$ in the uniform metric complete? Briefly justify your answer.

   [d] Let A be the set of elements in $[0,1]^N$ which are eventually 0. Determine the closure of A both in the product topology and in the uniform topology. Justify your answer.

5. Consider the 2-dimensional plane R^2 equipped with the equivalence relation $(x,y) \sim (x',y')$ if $xy = x'y'$ and consider the resulting quotient space $Y = R^2/\sim$.

   [a] Explain what we mean by the "resulting quotient space $Y = R^2/\sim$".

   [b] Show that Y is T1 meaning that all 1 point sets are closed.

   [c] Show that Y is homeomorphic to the real numbers.

6. **[a]** State what it means for two mappings from S^1 to a space X to be homotopic.

   [b] Let X be a topological space and $a \in X$. State what it means for two paths from a to a to be path-homotopic.

   [c] Let X be a topological space and $a \in X$. Let γ_1, γ_2 and γ_3 be three paths from a to a and let $*$ denote the usual concatenation of paths. Explain why it is not usually the case that $(\gamma_1 * \gamma_2) * \gamma_3 = \gamma_1 * (\gamma_2 * \gamma_3)$. Explain why this does not contradict the associativity property of the fundamental group.

   [d] Give an example of a topological space X, an element $a \in X$ and two paths from a to a which are homotopic but not path-homotopic. Explain your answer.

7. **[a]** Define what it means for a set A to be a retract of X.

   [b] If A is a retract of X and both are pathwise connected, show that the homomorphism induced by the inclusion map from A to X is injective.

   [c] Give an example of a retract for which the injective homomorphism from part **[c]** is not a bijection.

   [d] Define a deformation retract from X to A, state what it implies about the injective homomorphism from part **[c]** and give an example of a space where the above can be applied to compute the fundamental group. You can assume that the fundamental group of the circle is Z.

8. **[a]** Define a covering map. Consider the mapping from $(0, \infty)$ to S^1 given by $x \to e^{2\pi i x}$. Is this a covering map? Why or why not?

   [b] If X to A is a covering map, and f is a mapping from some space Y to A, define what a lifting of f is.

   [c] Give an example of a mapping f where a lifting does not exist and state some sufficient condition under which it does exist. Give a very brief explanation of why your sufficient condition is in fact sufficient.

   [d] Show that a covering map does not necessarily send a closed set to a closed set. It suffices to look at the standard covering map from the real numbers to the circle.